Poisson Algebras and 3D Superintegrable Hamiltonian Systems

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Abstract. Using a Poisson bracket representation, in 3D, of the Lie algebra $\mathfrak{sl}(2)$, we first use highest weight representations to embed this into larger Lie algebras. These are then interpreted as symmetry and conformal symmetry algebras of the "kinetic energy", related to the quadratic Casimir function. We then consider the potentials which can be added, whilst remaining integrable, leading to families of separable systems, depending upon arbitrary functions of a single variable. Adding further integrals, in the superintegrable case, restricts these functions to specific forms, depending upon a finite number of arbitrary parameters. The Poisson algebras of these superintegrable systems are studied. The automorphisms of the symmetry algebra of the kinetic energy are extended to the full Poisson algebra, enabling us to build the full set of Poisson relations.

Key words: Hamiltonian system; super-integrability; Poisson algebra; conformal algebra; constant curvature

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1   Introduction

This paper is in two parts. Sections 2–4 are mainly algebraic, building Lie algebras with a given copy of $\mathfrak{sl}(2)$ as a subalgebra. Since the second part of the paper (Sections 5 and 6) is about completely integrable Hamiltonian systems (and their *super-integrable* restrictions), the Lie algebraic part is presented in a Poisson bracket representation (with 3 degrees of freedom), so we are constructing Poisson algebras with linear relations. Our emphasis is on building a Poisson algebra with a desired Lie algebraic structure.

   We extend the 3-dimensional algebra $\mathfrak{sl}(2)$ to 6- and 10-dimensional algebras. The quadratic Casimir of the 6-dimensional algebra can be written in the form (with $n = 3$)

$$H_0 = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij}(\boldsymbol{q}) p_i p_j.$$

When the matrix of coefficients g^{ij} is nonsingular, it may be considered as the inverse of a metric tensor g_{ij} and the function H_0 represents the kinetic energy of a freely moving particle on the corresponding manifold (geodesic motion). For a metric with isometries, the infinitesimal generators (Killing vectors) correspond to functions which are *linear* in momenta and which Poisson commute with the kinetic energy H_0 (the corresponding Noether integrals). When the space is either flat or constant curvature, it possesses the maximal group of isometries, which is of dimension $\frac{1}{2}n(n+1)$. In this case, H_0 is actually the second order *Casimir* function of the symmetry algebra (see [10]). This is exactly the case we have, with $n = 3$ and a 6-dimensional

isometry algebra. The maximal number of conformal symmetries (including isometries as a sub-algebra) is of dimension $\frac{1}{2}(n+1)(n+2) = 10$, when $n = 3$. Our 10-dimensional extensions are just the corresponding conformal algebras. When g^{ij} is *singular*, the Poisson algebras have the same structure, but without the geometric interpretation.

Our main application of the algebraic structures we construct is to build some superintegrable systems with nontrivial, nonlinear Poisson algebras, which generalise the *Lie algebraic* Poisson algebras of Sections 2–4. Below we give a brief reminder of the meaning of complete and super-integrability.

A Hamiltonian system of n degrees of freedom, Hamiltonian H, is said to be *completely integrable in the Liouville sense* if we have n independent functions I_n, which are *in involution* (mutually Poisson commuting), with H being a function of these and typically just one of them. Whilst n is the maximal number of independent functions which can be *in involution*, it is possible to have further integrals of the Hamiltonian H, which necessarily generate a non-Abelian algebra of integrals of H. The maximal number of additional *independent* integrals is $n - 1$, since the "level surface" of $2n - 1$ integrals (meaning the intersection of individual level surfaces) is just the (unparameterised) integral curve. Well known elementary examples are the isotropic harmonic oscillator, the Kepler system and the Calogero–Moser system. The quadratures of complete integrability are often achieved through the separation of variables of the Hamilton–Jacobi equation. The solution of a maximally super-integrable system can also be calculated purely algebraically (albeit implicitly), requiring just the solution of the equations $I_k = c_k$, $k = 1, \dots, 2n - 1$. Maximally superintegrable systems have a number of interesting properties: they can be separable in more than one coordinate system; all bounded orbits are closed; they give rise to interesting Poisson algebras with polynomial Poisson relations. The idea can be extended to *quantum integrable systems*, with first integrals replaced by commuting differential operators. For some examples of superintegrable quantum systems it is possible to use the additional commuting operators to build sequences of eigenfunctions [7, 9]. There is a large literature on the classification and analysis of superintegrable systems (see the review [14]) and they naturally occur in many applications in physics (additional integrals being referred to as "hidden symmetries" [1]).

Clearly our geodesic flow, with Hamiltonian H_0 is super-integrable. There are 6 Noether integrals, but only 5 are functionally independent, since there is a quadratic constraint on the 6-dimensional algebra (see equation (3.3)). Furthermore, each element of the algebra commutes with at least one other element (see Table 1), so the Hamiltonian H_0 belongs to several involutive triples, each of which renders it completely integrable.

In Section 5 we show how to use the symmetry algebra of the kinetic energy H_0 to build quadratic (in momenta) integrals, and to add potential functions to build completely integrable systems, which are, in fact, separable. Explicitly, we extend the Hamiltonian functions H_0 to include a potential function:

$$H = H_0 + h(\boldsymbol{q}),$$

and use the symmetry algebra of H_0 to construct two functions

$$F_i = K_i + g_i(\boldsymbol{q}), \qquad \text{with} \qquad \{H, F_i\} = \{F_1, F_2\} = 0, \qquad i = 1, 2,$$

where the functions K_i are *quadratic forms* of the Noether constants of H_0. For some particular examples, we derive the explicit form of the functions h, g_i, which depend upon three arbitrary functions of a single variable (the three separation variables). Some examples are related through a Lie algebra automorphism, a property that would not be easy to see without the relation to the symmetry algebra of H_0.

In Section 6 we consider the superintegrable restrictions of these separable systems, by adding two further integrals. These now typically depend upon a small number of *arbitrary parameters*

instead of arbitrary functions. Whilst some of these functions still Poisson commute, not all of them can and the non-zero Poisson relations are no longer *linear*, but polynomial. In 3 degrees of freedom, these are considerably more complicated than in the 2 degrees of freedom case of [3, 7, 8], and it's not clear that we can always close the algebra in a finite way. However, since our kinetic energy has a 6-dimensional symmetry algebra with automorphisms which can be realised as canonical transformations, these act on the nonlinear Poisson algebra of our superintegrable systems, enabling us to obtain the full set of Poisson relations.

2   The basic setup

We consider a 6-dimensional space M, with (local) canonical coordinates q_i, p_i, $i = 1, 2, 3$, satisfying the standard canonical relations $\{q_i, q_j\} = \{p_i, p_j\} = 0$, $\{q_i, p_j\} = \delta_{ij}$, for all $i, j = 1, 2, 3$.

Recall that if f, g are any functions on M, then the Hamiltonian vector field of f is

$$X_f = \sum_{i=1}^{3} \left(\{q_i, f\}\partial_{q_i} + \{p_i, f\}\partial_{p_i} \right) \qquad \text{and} \qquad [X_f, X_g] = -X_{\{f,g\}}.$$

Functions which are *linear* in momenta define vector fields on configuration space, with coordinates (q_1, q_2, q_3). For any function on configuration space, $f(q_1, q_2, q_3)$, we have

$$h(\boldsymbol{q}, \mathbf{p}) = \sum_{i=1}^{3} a_i(\boldsymbol{q})p_i \quad \Rightarrow \quad \{f, h\} = \sum_{i=1}^{3} a_i(\boldsymbol{q})\frac{\partial f}{\partial q_i}. \tag{2.1}$$

Later, we use this to represent a Killing vector by its Noether constant, which is linear in momenta.

2.1   The Lie algebra $\mathfrak{g}_1 \simeq \mathfrak{sl}(2)$

We start with a Poisson representation of the Lie algebra $\mathfrak{sl}(2)$,

$$\begin{aligned}
&e_1 = p_2, \qquad h_1 = -2(q_1 p_1 + q_2 p_2 + q_3 p_3), \\
&f_1 = -2q_1 q_2 p_1 + \left(q_3^2 - q_1^2 - q_2^2\right)p_2 - 2q_2 q_3 p_3,
\end{aligned} \tag{2.2a}$$

satisfying

$$\{e_1, h_1\} = 2e_1, \qquad \{f_1, e_1\} = h_1, \qquad \{f_1, h_1\} = -2f_1. \tag{2.2b}$$

With this choice, the corresponding Hamiltonian vector fields will satisfy the standard commutation rules for $\mathfrak{sl}(2)$.

We can calculate the most general function on this phase space which commutes with the whole algebra $\mathfrak{g}_1$, which is a function of 3 variables:

Proposition 2.1 (general invariant of $\mathfrak{g}_1$)**.** *The most general function, I_1, on our phase space, satisfying $\{e_1, I_1\} = \{h_1, I_1\} = \{f_1, I_1\} = 0$, is given by*

$$I_1 = F(r_0, r_1, r_2), \tag{2.3}$$

with

$$r_0 = \frac{q_3}{q_1}, \qquad r_1 = q_3 p_1 + q_1 p_3, \qquad r_2 = -\left(q_1^2 + q_3^2\right)p_1^2 + q_1^2 p_2^2 - 2q_1 q_3 p_1 p_3,$$

where F is an arbitrary function of 3 variables.

In particular, the Casimir function is given by

$$\mathcal{C}_1 = e_1 f_1 + \frac{1}{4} h_1^2 = r_0^2 r_1^2 + (r_0^2 - 1) r_2. \tag{2.4}$$

Remark 2.2. In 2-dimensions, *all* invariants would be just functions of the quadratic Casimir, but in this larger space the general invariant includes all the Casimirs of larger algebras containing $\mathfrak{g}_1$ as a subalgebra.

The most general *quadratic* (in momenta) function of the form (2.3) is given by

$$
\begin{aligned}
I_2 &= \psi(r_0) r_1^2 - \varphi(r_0) r_2 \\
&= \varphi\left(\frac{q_3}{q_1}\right) q_1^2 (p_1^2 - p_2^2 - p_3^2) + \left(\varphi\left(\frac{q_3}{q_1}\right) + \psi\left(\frac{q_3}{q_1}\right)\right) (q_3 p_1 + q_1 p_3)^2.
\end{aligned} \tag{2.5}
$$

The determinant of the matrix of coefficients, G^{ij}, is $\det G = \varphi^2 q_1^6 \big(\varphi r_0^2 + (r_0^2 - 1)\psi\big)$. When this is nonzero, G defines a *conformally flat* metric, but the Ricci scalar is generally a complicated differential expression in the functions φ and ψ, even in the diagonal case, for which $\psi = -\varphi$. In the diagonal case there are two interesting cases:

1) $\varphi = (c_1 r_0 + c_2)^2$, which is a constant curvature space with $R = 6(c_1^2 - c_2^2)$,

2) $\varphi = c_1(r_0^2 - 1)$, which is *not* a constant curvature space, but does have constant scalar curvature $R = 2c_1$.

In Section 5.3 we consider the involutive system

$$H = H_0 + h, \qquad F_1 = \mathcal{C}_1 + g_1, \qquad F_2 = K^2 + g_2,$$

where $H_0 = I_2$ is *conformally flat* (but not constant curvature) and K is some element of $\mathfrak{g}_1$, to determine separable choices of potential function.

2.1.1 Lie algebra automorphisms as canonical transformations

We can build the standard Lie algebra automorphisms of $\mathfrak{sl}(2)$ as canonical transformations. We denote by ι_1 and ι_2, the *involutive automorphisms*

$$\iota_1: (e_1, h_1, f_1) \mapsto (f_1, -h_1, e_1), \tag{2.6a}$$

$$\iota_2: (e_1, h_1, f_1) \mapsto (-e_1, h_1, -f_1), \tag{2.6b}$$

which can be realised by canonical transformations, generated by

$$S_1 = \frac{q_1 P_1 - q_2 P_2 + q_3 P_3}{q_1^2 - q_2^2 - q_3^2}, \qquad S_2 = q_1 P_1 - q_2 P_2 + q_3 P_3. \tag{2.6c}$$

Each of the functions r_0, r_1, r_2 of (2.3) is invariant under these automorphisms.

2.2 Our choice of $\mathfrak{sl}(2)$

The calculations of this paper could be carried out for any choice of representation of $\mathfrak{sl}(2)$. Perhaps the most natural choice would be the *linear representation*

$$e_1 = 2(Q_1 P_2 + Q_2 P_3), \qquad h_1 = 2(Q_1 P_1 - Q_3 P_3), \qquad f_1 = Q_2 P_1 + Q_3 P_2, \tag{2.7}$$

which is related to the representation (2.2a) through the point transformation

$$Q_1 = \frac{1}{q_1}, \qquad Q_2 = \frac{2q_2}{q_1}, \qquad Q_3 = \frac{2(q_2^2 + q_3^2 - q_1^2)}{q_1}.$$

However, the first step in our calculation of Section 2.3 is to seek functions which *commute* with e_1, so it is natural to transform e_1 to p_i for some i. The invariants of e_1 are Q_1 and $2Q_1Q_3 - Q_2^2$, and we have $\left\{\frac{Q_2}{2Q_1}, e_1\right\} = 1$, so we initially choose

$$q_1 = \rho(Q_1), \qquad q_2 = \frac{Q_2}{2Q_1}, \qquad q_3 = \Theta(Q_1, 2Q_1Q_3 - Q_2^2),$$

which imply

$$\{q_1, h_1\} = 2Q_1\rho'(Q_1), \qquad \{q_2, h_1\} = -2q_2, \qquad \{q_3, h_1\} = 2Q_1\Theta_1(Q_1, 2Q_1Q_3 - Q_2^2),$$

where $\Theta_1(y_1, y_2)$ is the partial derivative of $\Theta(y_1, y_2)$ with respect to y_1.

If we *choose* to have a symmetric formula for h_1, so that $\{q_i, h_1\} = -2q_i$, then

$$q_1 = \frac{1}{Q_1}, \qquad q_3 = \frac{\sigma(2Q_1Q_3 - Q_2^2)}{Q_1},$$

with q_3 defined up to an arbitrary function of *one* variable only. The inverse of this transformation is just

$$Q_1 = \frac{1}{q_1}, \qquad Q_2 = \frac{2q_2}{q_1}, \qquad Q_3 = \frac{1}{2}q_1\left(\frac{4q_2^2}{q_1^2} + \sigma^{-1}\left(\frac{q_3}{q_1}\right)\right),$$

and the corresponding canonical transformation, with (2.7), gives

$$e_1 = p_2, \qquad h_1 = -2(q_1p_1 + q_2p_2 + q_3p_3),$$
$$f_1 = -2q_1q_2p_1 + \left(\frac{1}{4}\sigma^{-1}\left(\frac{q_3}{q_1}\right)q_1^2 - q_2^2\right)p_2 - 2q_2q_3p_3, \tag{2.8a}$$

which is exactly (2.2a) when $\sigma^{-1}(r_0) = 4(r_0^2 - 1)$, so $\sigma(y_2) = \frac{1}{2}\sqrt{y_2 + 4}$.

In fact, given this choice of e_1, h_1, the most general form of f_1 is determined only up to 3 *arbitrary functions*:

$$f_1 = \left(uq_1^2 - 2q_1q_2\right)p_1 + \left(vq_1^2 - q_2^2\right)p_2 + \left(wq_1^2 - 2q_2q_3\right)p_3, \tag{2.8b}$$

u, v, w being arbitrary functions of $\frac{q_3}{q_1}$. Our transformed elements (2.8a) just correspond to $u = w = 0, v = \frac{1}{4}\sigma^{-1}$.

Remark 2.3. Lie classified the 2-dimensional realisations of $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{R})$. There are 5 inequivalent realisations of $\mathfrak{sl}(2, \mathbb{R})$ (see [2, Section 2]). No such classification exists for 3-dimensional realisations, but not all choices of u, v, w in (2.8b) lead to equivalent realisations. For example, the determinant of the matrix of coefficients in (2.4) (for general u, v, w) vanishes when $q_1w\left(\frac{q_3}{q_1}\right) - q_3u\left(\frac{q_3}{q_1}\right) = 0$ (as in our case) and this cannot be equivalent to a non-degenerate case.

2.3 Highest weight representations of $\mathfrak{g}_1$

We now build highest weight representations of $\mathfrak{g}_1$, starting with

$$Z_1 = A(q_1, q_2, q_3)p_1 + B(q_1, q_2, q_3)p_2 + C(q_1, q_2, q_3)p_3$$

and requiring (2.9b) below, which leads immediately to

$$Z_1 = q_1^{1-\frac{\lambda}{2}}\left(A\left(\frac{q_3}{q_1}\right)p_1 + B\left(\frac{q_3}{q_1}\right)p_2 + C\left(\frac{q_3}{q_1}\right)p_3\right), \tag{2.9a}$$

satisfying

$$\{Z_1, e_1\} = 0, \qquad \{Z_1, h_1\} = \lambda Z_1. \tag{2.9b}$$

Defining

$$Z_{i+1} = \{Z_i, f_1\}, \qquad i = 1, 2, \ldots, \tag{2.9c}$$

the Poisson relations (2.2b) then imply

$$\{Z_i, h_1\} = (\lambda - 2i + 2)Z_i \qquad \text{and} \qquad \{Z_i, e_1\} = (i-1)(\lambda - i + 2)Z_{i-1}. \tag{2.9d}$$

From this point, A, B and C are functions of $r_0 = \frac{q_3}{q_1}$.

For general λ, we have an infinite-dimensional representation, but when $\lambda = 2m$ (m a positive integer), it is finite, with dimension $2m + 1$. We are particularly interested in the cases $m = 0, 1$.

1. *The case $m = 0$:* It is easy to see that the general formula for Z_2 is given by

$$Z_2 = q_1^{1-\frac{\lambda}{2}}\left((\lambda q_2 A + 2q_1 B)p_1 + (\lambda q_2 B + 2q_1 A - 2q_3 C)p_2 + (\lambda q_2 C + 2q_3 B)p_3\right),$$

so that, for $Z_2 = 0$, we require $\lambda = 0$, $A = \frac{q_3}{q_1}C$ and $B = 0$. This leads to

$$Z_1 = C\left(\frac{q_3}{q_1}\right)(q_3 p_1 + q_1 p_3) \qquad \text{and} \qquad \{Z_1, e_1\} = \{Z_1, f_1\} = \{Z_1, h_1\} = 0. \tag{2.10}$$

In the notation of (2.3), $Z_1 = C(r_0)r_1$.

2. *The case $m = 1$:* For $m \geq 1$ and $\lambda = 2m$, we automatically have $Z_{2m+2} = 0$, without any restrictions on the functions A, B, C. When $m = 1$, we have

$$\begin{aligned}
Z_1 &= Ap_1 + Bp_2 + Cp_3, \\
Z_2 &= 2(Aq_2 + Bq_1)p_1 + 2(Aq_1 + Bq_2 - Cq_3)p_2 + 2(Cq_2 + Bq_3)p_3, \\
Z_3 &= 2\left(A\left(q_1^2 + q_2^2 + q_3^2\right) + 2Bq_1q_2 - 2Cq_1q_3\right)p_1 + 2\left(B\left(q_1^2 + q_2^2 - q_3^2\right)\right. \\
&\quad \left. + 2Aq_1q_2 - 2Cq_2q_3\right)p_2 + 2\left(C\left(q_2^2 - q_1^2 - q_3^2\right) + 2Aq_1q_3 + 2Bq_2q_3\right)p_3.
\end{aligned} \tag{2.11}$$

The Poisson bracket relations (2.9c) and (2.9d) take the explicit form

$$\begin{aligned}
&\{Z_1, h_1\} = 2Z_1, \qquad \{Z_1, f_1\} = Z_2, \qquad \{Z_2, e_1\} = 2Z_1, \qquad \{Z_2, f_1\} = Z_3, \\
&\{Z_3, e_1\} = 2Z_2, \qquad \{Z_3, h_1\} = -2Z_3, \qquad \{Z_1, e_1\} = \{Z_2, h_1\} = \{Z_3, f_1\} = 0.
\end{aligned} \tag{2.12}$$

3 Building Poisson algebras

We have a Lie algebra $\mathfrak{g}_1$ and its action (through the Poisson bracket) on the representation space $\{Z_i\}_{i=1}^{2m+1}$. We may consider the *linear space* spanned by $\mathfrak{g}_1$ and $\mathfrak{g}_2$:

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2, \qquad \text{where} \qquad \mathfrak{g}_2 = \{Z_i\}_{i=1}^{2m+1} \qquad \text{and} \qquad \{\mathfrak{g}_1, \mathfrak{g}_1\} \subset \mathfrak{g}_1, \qquad \{\mathfrak{g}_1, \mathfrak{g}_2\} \subset \mathfrak{g}_2,$$

but need to specify the possible forms of $\{\mathfrak{g}_2, \mathfrak{g}_2\}$ if we wish to consider $\mathfrak{g}$ as a Lie algebra.

Using this approach, we now build 6-dimensional algebras. The quadratic Casimir function of the algebra $\mathfrak{g}$ defines a matrix, which can be interpreted as an upper-index metric, when it is non-singular. In this case its inverse defines a metric with Killing vectors corresponding to the elements of $\mathfrak{g}$.

The calculation splits into two parts. First of all we use the Jacobi identity to derive abstract relations. Then, in Section 3.2 we use these relations to restrict the functions in the concrete realisation of (2.11).

In Section 4, we further extend to 10-dimensional algebras, which can be interpreted as conformal symmetry algebras of these metrics.

3.1 6-dimensional extensions

If we consider $\mathfrak{g}_2$ to have the basis Z_1, Z_2, Z_3, defined by (2.11), then it follows from the Poisson bracket relations (2.12), that

$$\{Z_1, h_1\} = 2Z_1, \qquad \{Z_2, h_1\} = 0, \qquad \{Z_3, h_1\} = -2Z_3,$$

so, for this 3-dimensional invariant space we introduce the notation

$$e_2 = Z_1, \qquad h_2 = Z_2, \qquad f_2 = Z_3.$$

For $\mathfrak{g}$ to form a Lie algebra, we must have

$$\{\mathfrak{g}_2, \mathfrak{g}_2\} \subset \mathfrak{g}_1 + \mathfrak{g}_2,$$

including the special case when $\mathfrak{g}_2$ is *Abelian*.

Noting that $\{\{e_2, h_2\}, h_1\} = 2\{e_2, h_2\}$, we have

$$\{e_2, h_2\} = \alpha e_1 + \beta e_2, \qquad \text{for arbitrary constants } \alpha, \ \beta.$$

The action of f_1 leads to

$$\{e_2, f_2\} = -\alpha h_1 + \beta h_2, \qquad \text{and} \qquad \{h_2, f_2\} = -2\alpha f_1 + \beta f_2.$$

In fact, we may choose $\beta = 0$ without loss of generality, as shown by the following:

Proposition 3.1 ($\{\mathfrak{g}_2, \mathfrak{g}_2\} \subset \mathfrak{g}_1$). *If the vector space $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ forms a Lie algebra, satisfying the Poisson bracket relations (2.2b) and (2.12), then a basis can be chosen for $\mathfrak{g}_2$, satisfying $\{\mathfrak{g}_2, \mathfrak{g}_2\} \subset \mathfrak{g}_1$. Specifically, there exists a parameter a, such that*

$$\{e_2, h_2\} = ae_1, \qquad \{e_2, f_2\} = -ah_1, \qquad \{h_2, f_2\} = -2af_1. \tag{3.1}$$

The possibility of $a = 0$ is included, in which case $\mathfrak{g}_2$ forms an Abelian subalgebra.

Proof. Defining

$$\hat{e}_2 = e_2 + \gamma e_1 \quad \Rightarrow \quad \hat{h}_2 = h_2 - \gamma h_1 \qquad \text{and} \qquad \hat{f}_2 = f_2 - 2\gamma f_1,$$

for arbitrary parameter γ, then

$$\{\hat{e}_2, \hat{h}_2\} = (\alpha - 2\gamma^2)e_1 + (\beta - 4\gamma)e_2.$$

Choosing $\gamma = \frac{1}{4}\beta$, we have

$$\{\hat{e}_2, \hat{h}_2\} = \hat{\alpha}e_1, \qquad \text{where} \qquad \hat{\alpha} = \alpha - \frac{1}{8}\beta^2.$$

The action of f_1 then leads to

$$\{\hat{e}_2, \hat{f}_2\} = -\hat{\alpha}h_1, \qquad \text{and} \qquad \{\hat{h}_2, \hat{f}_2\} = -2\hat{\alpha}f_1,$$

giving (3.1) after dropping "hats" and setting $\hat{\alpha} = a$. ∎

Casimir functions. This 6-dimensional algebra has a quadratic Casimir function

$$\mathcal{C}_{12} = 2a\left(e_1 f_1 + \frac{1}{4}h_1^2\right) + 2e_2 f_2 - h_2^2, \tag{3.2}$$

which will play an important role in what follows.

As an abstract (rank 2) algebra, there is a second independent (fourth order) Casimir element

$$\mathcal{C}_4 = \left(e_1 f_2 + f_2 e_1 + h_1 h_2 + h_2 h_1 - 2(f_1 e_2 + e_2 f_1)\right)^2.$$

In the 6×6 matrix representation (the adjoint representation), this is a multiple of the identity matrix, but in our Poisson representation, it vanishes identically, so the 6-dimensional Poisson algebra has a quadratic constraint:

$$e_1 f_2 + h_1 h_2 - 2f_1 e_2 = 0. \tag{3.3}$$

3.2 The 3 non-Abelian subcases of $\{\mathfrak{g}_2, \mathfrak{g}_2\} \subset \mathfrak{g}_1$

The relations (3.1) impose conditions on the functions A, B, C, giving 3 subcases:

1) $A(r_0) = \sqrt{\frac{a}{2(r_0^2 - 1)}}$, $B(r_0) = 0$, $C(r_0) = r_0 A(r_0)$,

2) $A(r_0) = 0$, $B(r_0) = \sqrt{\frac{-a}{2}}$, $C(r_0) = 0$, for $a < 0$,

3) $B(r_0) = 0$, with $A(r_0)$ and $C(r_0)$ satisfying the equation

$$A'(r_0) - r_0 C'(r_0) = \frac{2A^2(r_0) - 2C^2(r_0) + a}{2r_0 A(r_0) - 2C(r_0)}, \qquad \text{with} \qquad C(r_0) \neq r_0 A(r_0). \qquad (3.4)$$

3.2.1 Case 1:

Here we have the explicit solution (given here for $a = 2$):

$$e_2 = \frac{q_1 p_1 + q_3 p_3}{\sqrt{q_3^2 - q_1^2}}, \qquad h_2 = \frac{2q_2(q_1 p_1 + q_3 p_3) - 2(q_3^2 - q_1^2)p_2}{\sqrt{q_3^2 - q_1^2}},$$
$$f_2 = \frac{2(q_1^2 + q_2^2 - q_3^2)(q_1 p_1 + q_3 p_3) - 4q_2(q_3^2 - q_1^2)p_2}{\sqrt{q_3^2 - q_1^2}}.$$

In this case

$$2e_2 f_2 - h_2^2 = -4\left(e_1 f_1 + \frac{1}{4}h_1^2\right),$$

so the Casimir (3.2) vanishes, corresponding to a quadratic constraint between the basis elements.

The most general invariant of this 6-dimensional algebra is a restriction of (2.3), given by $I_1 = F(r_0, r_1)$, with the most general quadratic invariant being

$$H = \psi(r_0)r_1^2 = \psi\left(\frac{q_3}{q_1}\right)(q_3 p_1 + q_1 p_3)^2, \qquad \text{with arbitrary function } \psi.$$

3.2.2 Case 2:

This just leads to the trivial case $\mathfrak{g}_2 = \mathfrak{g}_1$.

3.2.3 Case 3:

This is the most interesting case, depending on two arbitrary functions, subject to one differential constraint (3.4). The explicit form of the Casimir (3.2) is

$$H = 2\left(2(q_1 A - q_3 C)^2 + a\left(q_1^2 - q_3^2\right)\right)\left(p_1^2 - p_2^2 - p_3^2\right)$$
$$+ 2\left(a + 2\left(A^2 - C^2\right)\right)(q_3 p_1 + q_1 p_3)^2, \qquad (3.5)$$

which is a specific example of the general quadratic integral (of $\mathfrak{g}_1$), given in (2.5).

Remark 3.2 (constant curvature). When

$$C(r_0) \neq r_0 A(r_0) \qquad \text{and} \qquad 2(A(r_0) - r_0 C(r_0))^2 \neq a\left(r_0^2 - 1\right),$$

then the matrix of coefficients is invertible and defines a metric with constant curvature, satisfying

$$R_{ij} = \frac{1}{n}R g_{ij}, \qquad (3.6)$$

where, in our case $n = 3$ and $R = -12a$.

The six first degree (in momenta) Hamiltonian functions generate six Killing vectors (by the formula (2.1)) of the metric corresponding to the Hamiltonian (3.5). The Poisson algebra is given by Table 1.

Table 1. The 6-dimensional symmetry algebra $\mathfrak{g}$, when $\{\mathfrak{g}_2, \mathfrak{g}_2\} \subset \mathfrak{g}_1$.

	e_1	h_1	f_1	e_2	h_2	f_2
e_1	0	$2e_1$	$-h_1$	0	$-2e_2$	$-2h_2$
h_1	$-2e_1$	0	$2f_1$	$-2e_2$	0	$2f_2$
f_1	h_1	$-2f_1$	0	$-h_2$	$-f_2$	0
e_2	0	$2e_2$	h_2	0	ae_1	$-ah_1$
h_2	$2e_2$	0	f_2	$-ae_1$	0	$-2af_1$
f_2	$2h_2$	$-2f_2$	0	ah_1	$2af_1$	0

The Lie algebra automorphisms ι_1 and ι_2

The automorphisms of $\mathfrak{g}_1$, given by (2.6) also act on this extended algebra:

$$\iota_1\colon\ (e_1, h_1, f_1, e_2, h_2, f_2) \mapsto \left(f_1, -h_1, e_1, -\frac{1}{2}f_2, -h_2, -2e_2\right),$$

$$\iota_2\colon\ (e_1, h_1, f_1, e_2, h_2, f_2) \mapsto (-e_1, h_1, -f_1, e_2, -h_2, f_2),$$

with the Casimir function (3.5) being invariant (it being a function of the invariants r_0, r_1 and r_2).

3.3 The case when $\mathfrak{g}_2$ is Abelian

When $\mathfrak{g}_2$ is an Abelian algebra, we have

$$\{e_2, h_2\} = 0, \qquad \{e_2, f_2\} = 0, \qquad \{h_2, f_2\} = 0,$$

so we have the Poisson algebra of Table 1, but with $a = 0$, giving a 3×3 block of zeros.

In this case the Killing form of the 6-dimensional algebra $\mathfrak{g}$ is degenerate, but the Casimir can be obtained by taking the limit of (3.2) as $a \to 0$, giving

$$H = 2e_2 f_2 - h_2^2. \tag{3.7}$$

As with Case 3, above, we have $B(r_0) = 0$ and the functions $A(r_0)$ and $C(r_0)$ satisfy the differential constraint

$$A'(r_0) - r_0 C'(r_0) = \frac{A^2(r_0) - C^2(r_0)}{r_0 A(r_0) - C(r_0)}, \tag{3.8}$$

which is just (3.4) with $a = 0$.

The explicit form of the Casimir (3.7) is given by

$$H = 4(q_1 A - q_3 C)^2 \left(p_1^2 - p_2^2 - p_3^2\right) + 4\left(A^2 - C^2\right)(q_3 p_1 + q_1 p_3)^2, \tag{3.9}$$

which is just (3.5), with $a = 0$, and non-degenerate when

$$(A - r_0 C)(C - r_0 A) \neq 0, \tag{3.10}$$

in which case it corresponds to a *flat* metric when the functions A and C satisfy (3.8).

3.4 The solutions of (3.4) and (3.8)

In Sections 3.2 and 3.3, we gave two classes of Poisson algebra $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, with Casimir functions (3.5) and (3.9), corresponding (when non-degenerate) to constant curvature and flat spaces, respectively. These depend on 2 functions $A(r_0)$ and $C(r_0)$, which must satisfy the differential relations (3.4) or (3.8) respectively. In this section we consider the general solution of these equations and some particular cases of interest.

The general solution is constructed in two steps. First we reduce the problem to finding only one function $A(r_0)$, with $C(r_0) = 1$ or $C(r_0) = 0$. The second step reintroduces the second function.

3.4.1 The solutions of (3.4)

First, we note that Z_i of (2.11) are only defined up to an overall multiple of a function of r_0, since this is an invariant of the algebra $\mathfrak{g}_1$. Therefore, to satisfy (2.11), we have two cases

1) $C(r_0) \neq 0$, in which case we may set $C(r_0) = 1$ and then determine the *one* function $A(r_0)$,

2) $C(r_0) = 0 \Rightarrow A(r_0) = \frac{1}{2}\sqrt{c_1 r_0^2 - 2a}$.

For the case $C(r_0) = 1$, (3.4) takes the form

$$A'(r_0) = \frac{2A^2(r_0) - 2 + a}{2(r_0 A(r_0) - 1)}, \qquad \text{with} \qquad r_0 A(r_0) \neq 1.$$

We then have a number of subcases.

When $A'(r_0) \neq 0$, we have the general solution

$$A = \frac{(a-2)\big(2r_0 + c_1\sqrt{2(a-2)}\sqrt{2 - (a-2)^2 c_1^2 + (a-2)r_0^2}\big)}{2c_1^2(a-2)^2 - 4}, \qquad \text{when } a \neq 2, \qquad (3.11a)$$

$$A = \frac{1}{r_0 \pm \sqrt{r_0^2 - 2c_1}} = \frac{r_0 \mp \sqrt{r_0^2 - 2c_1}}{2c_1}, \qquad \text{when } a = 2. \qquad (3.11b)$$

When $A'(r_0) = 0$, then

$$A(r_0) = \sqrt{\frac{2-a}{2}}. \qquad (3.11c)$$

Clearly, when $(a-2)^2 c_1^2 - 2 = 0$ the solution (3.11a) is singular. Replacing c_1 by $c_2 = (a-2)^2 c_1^2 - 2$, we find

$$\big(2A^2 + a - 2\big)c_2 = (a-2)\big(4r_0 A + (a-2)r_0^2 - 2\big)$$

$$\Rightarrow \quad A = \frac{(2-a)r_0^2 + 2}{4r_0}, \qquad \text{when } c_2 = 0. \qquad (3.12a)$$

On the other hand, when $c_1 = 0$, we have the simple solution

$$A = \frac{(2-a)r_0}{2}. \qquad (3.12b)$$

The first form of (3.11b) allows us to set $c_1 = 0$ (with the "+" sign) to obtain the special solution

$$A = \frac{1}{2r_0}. \qquad (3.12c)$$

3.4.2   The solutions of (3.8)

The solution of (3.8) is just a reduction of those of (3.4), but with $a = 0$, giving

$$A = \frac{r_0 + c_1\sqrt{r_0^2 + c_1^2 - 1}}{1 - c_1^2}, \tag{3.13a}$$

$$A = \frac{r_0^2 + 1}{2r_0}, \tag{3.13b}$$

$$A = r_0, \tag{3.13c}$$

$$A = 1, \tag{3.13d}$$

which are respectively reductions of (3.11a), (3.12a), (3.12b) and (3.11c).

3.4.3   Reinstating the second function

We can now reinstate the second function by writing

$$Z_1 = \sigma(r_0)(A(r_0)p_1 + p_3), \tag{3.14}$$

where $A(r_0)$ is one of the solutions (3.11) or (3.12). The conditions (3.1) then imply

$$\frac{2\sigma\sigma'}{\sigma^2 - 1} = \frac{a}{(r_0 - A)(r_0 A - 1)}, \tag{3.15}$$

which can be directly integrated for a given solution $A(r_0)$.

   We see from (3.15) that when $a = 0$, we generically have $\sigma' = 0$, so can just multiply the solutions (3.13) by an arbitrary constant. There is a singular solution of (3.15): $A = r_0$ and σ *arbitrary.*

3.5   The Casimirs (3.5) and (3.9) for some specific solutions

The general formulae for the Casimirs (3.5) and (3.9) depend upon the specific functions A and C. For any solution given in Section 3.4, we can calculate the specific form of H_0 (the corresponding kinetic energy). Each one corresponds to a constant curvature or flat manifold, so will not all be independent. In fact, all constant curvature metrics with the same dimension, signature and scalar curvature R are isometrically related (see [5, p. 84]). Since, in our case, we have $R = -12a$, any two cases with the same value of a should be isometric, even though the transformation may be difficult to find.

3.5.1   The Hamiltonian for case (3.11c)

For this choice, (3.5) takes the form

$$H_0 = 2\big(\sqrt{2}q_1 - \sqrt{2-a}\,q_3\big)^2\big(p_1^2 - p_2^2 - p_3^2\big) \tag{3.16a}$$

$$= 4q_1^2\big(p_1^2 - p_2^2 - p_3^2\big), \qquad \text{when} \quad a = 2. \tag{3.16b}$$

This restriction of $a = 2$ corresponds to $A = 0$, and gives the 6-dimensional isometry algebra

$$e_1 = p_2, \qquad h_1 = -2(q_1p_1 + q_2p_2 + q_3p_3),$$
$$f_1 = -2q_1q_2p_1 + \big(q_3^2 - q_1^2 - q_2^2\big)p_2 - 2q_2q_3p_3,$$
$$e_2 = p_3, \qquad h_2 = 2(q_2p_3 - q_3p_2),$$
$$f_2 = -4q_3(q_1p_1 + q_2p_2) - 2\big(q_1^2 - q_2^2 + q_3^2\big)p_3, \tag{3.16c}$$

which satisfies the relations of Table 1 for $a = 2$. This will be embedded into the 10-dimensional algebra (4.3) in Section 4.3.2 and will be one of our main examples in the context of super-integrability in Section 6.

3.5.2 The Hamiltonian for case (3.12a)

For this choice, (3.5) takes the form

$$H_0 = \frac{\left(2q_1^2 + (a-2)q_3^2\right)^2}{4q_1^2 q_3^2}\left((q_1^2 + q_3^2)p_1^2 - q_1^2 p_2^2 + 2q_1 q_3 p_1 p_3\right),$$

which simplifies with the reduction $a = 2$ and also reduces to the flat case, with $a = 0$.

3.5.3 The Hamiltonian for case (3.12b)

For this choice, (3.5) takes the form

$$H_0 = \left(2q_1^2 + (a-2)q_3^2\right)\left(a\left(p_1^2 - p_2^2 - p_3^2\right) + \left(\frac{a-2}{q_1^2}\right)\left(q_3 p_1 + q_1 p_3\right)^2\right)$$
$$= 4q_1^2\left(p_1^2 - p_2^2 - p_3^2\right), \qquad \text{when} \quad a = 2.$$

This restriction to $a = 2$ is identical to (3.16b), so corresponds to the same algebra (3.16c).

3.5.4 The Hamiltonian for case (3.13d)

The flat case (3.13d) is just the case (3.11c), with $a = 0$, so $A = 1$. However, we saw that when $a = 0$, equation (3.15) has a constant solution, so we make the choice $\sigma = \frac{1}{2}$, in which case, the Casimir (3.9) takes the form

$$H_0 = (q_1 - q_3)^2\left(p_1^2 - p_2^2 - p_3^2\right). \tag{3.17a}$$

The 6-dimensional isometry algebra now takes the form

$$\begin{aligned}
&e_1 = p_2, \qquad h_1 = -2(q_1 p_1 + q_2 p_2 + q_3 p_3), \\
&f_1 = -2q_1 q_2 p_1 + \left(q_3^2 - q_1^2 - q_2^2\right)p_2 - 2q_2 q_3 p_3, \\
&e_2 = \frac{1}{2}(p_1 + p_3), \qquad h_2 = q_2(p_1 + p_3) + (q_1 - q_3)p_2, \\
&f_2 = \left(q_2^2 + (q_1 - q_3)^2\right)p_1 + 2q_2(q_1 - q_3)p_2 + \left(q_2^2 - (q_1 - q_3)^2\right)p_3,
\end{aligned} \tag{3.17b}$$

which satisfies the relations of Table 1 for $a = 0$. This will be embedded into the 10-dimensional algebra (4.7) in Section 4.4.2 and will be one of our main examples in the context of superintegrability in Section 6.

Flat coordinates. Since e_2, h_2, f_2 are in involution, we can consider them as new *momenta*, $P_1 = e_2$, $P_2 = h_2$, $P_3 = f_2$, and find new coordinates Q_i, which are canonically conjugate. This is just Lie's theorem on complete integrability in the Poisson case. The equations $\{Q_i, P_j\} = \delta_{ij}$ give us a system of equations for Q_i, which in the current case are easy to solve:

$$Q_1 = \frac{q_1^2 - q_2^2 - q_3^2}{q_1 - q_3}, \qquad Q_2 = \frac{q_2}{q_1 - q_3}, \qquad Q_3 = \frac{-1}{2(q_1 - q_3)}. \tag{3.18}$$

With generating function $S = \frac{(q_1^2 - q_2^2 - q_3^2)P_1 + q_2 P_2 - \frac{1}{2}P_3}{q_1 - q_3}$, we then have

$$\begin{aligned}
&e_1 = -2(Q_2 P_1 + Q_3 P_2), \qquad h_1 = 2(Q_3 P_3 - Q_1 P_1), \qquad f_1 = -Q_1 P_2 - Q_2 P_3, \\
&e_2 = P_1, \qquad h_2 = P_2, \qquad f_2 = P_3,
\end{aligned}$$

leading to

$$H_0 = 2P_1 P_3 - P_2^2. \tag{3.19}$$

The form of this is dictated by the form of the Casimir (3.7). It can, of course, be diagonalised to $H_0 = 2P_1^2 - P_2^2 - 2P_3^2$ by using $Q_1 \pm Q_3$.

4   Extending to the conformal algebra

In Section 3 we built 6-dimensional Poisson algebras which included $\mathfrak{g}_1$ as a subalgebra. The quadratic Casimir function was interpreted as a Hamiltonian function (the kinetic energy), with the algebra $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ being its *symmetry algebra*. When the matrix of coefficients was non-degenerate, this defined a metric, and the symmetry algebra corresponded to its *Killing vectors*. In this section we further extend the algebra $\mathfrak{g}$ to include conformal symmetries, which, in the metric case, correspond to *conformal Killing vectors*. In fact, we will first construct an extension with the appropriate Poisson bracket relations and then prove directly that these are conformal symmetries of the above Hamiltonian.

4.1   Conformal algebras

In 2 dimensions, as is well known, the conformal group is *infinite*. For $n \geq 3$ this group is *finite* and has *maximal* dimension $\frac{1}{2}(n+1)(n+2)$, which is achieved for *conformally flat* spaces (which includes *flat* and *constant curvature* spaces). We are particularly interested in the case $n = 3$, so will be looking for a 10-dimensional algebra.

In flat spaces, the infinitesimal generators consist of n *translations*, $\frac{1}{2}n(n-1)$ *rotations*, 1 *scaling* and n *inversions*, totalling $\frac{1}{2}(n+1)(n+2)$. This algebra is isomorphic to $\mathfrak{so}(n+1, 1)$ (see [4, p. 143]).

For this discussion, we distinguish between "true symmetries", which we label X_s, and "conformal symmetries", which we label X_c. The "true symmetries" form a subalgebra of the conformal symmetry algebra. Here we discuss the general structure of the conformal algebra.

Suppose X_s is a symmetry and X_{c1}, X_{c2} are conformal symmetries of H, satisfying

$$\{X_s, H\} = 0, \qquad \{X_{ci}, H\} = w_i H,$$

where w_i are functions of the coordinates q_1, q_2, q_3. Then the Jacobi identity implies:

$$\{\{X_s, X_{ci}\}, H\} = -\{w_i, X_s\}H, \qquad \{\{X_{c1}, X_{c2}\}, H\} = (\{w_1, X_{c2}\} - \{w_2, X_{c1}\})H.$$

The symmetry X_s is, of course, just a special conformal symmetry, with $w = 0$. Whilst it may be that $\{w_i, X_s\} = 0$ for some particular choices of X_s or X_{ci} and that $(\{w_1, X_{c2}\} - \{w_2, X_{c1}\})$ may or may not be zero, these relations show that conformal symmetries form an invariant space under the action of the "true" symmetries and that the set of conformal symmetries (including the "true" symmetries) form a Lie algebra. In particular, the conformal symmetries must form an invariant space under the action of $\mathfrak{g}_1$.

4.2   Building the additional elements

We start with the 6-dimensional Lie algebra $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, where $\mathfrak{g}_2$ is either Case 3 of Section 3.2 or the Abelian case of Section 3.3. The respective Casimir functions H correspond to a space with non-zero, constant curvature and a space of zero curvature.

We first *algebraically* extend $\mathfrak{g}$ by adding a further 4 basis elements, so that, as a vector space, we have

$$\hat{\mathfrak{g}} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3 + \mathfrak{g}_4,$$

where $\mathfrak{g}_3$ is another 3-dimensional invariant space in the form of either Case 3 of Section 3.2 or the Abelian case of Section 3.3, and $\mathfrak{g}_4$ is a 1-dimensional representation of the form (2.10). We already know the bracket relations

$$\{\mathfrak{g}_1, \mathfrak{g}_1\}, \qquad \{\mathfrak{g}_2, \mathfrak{g}_2\}, \qquad \{\mathfrak{g}_3, \mathfrak{g}_3\}, \qquad \{\mathfrak{g}_1, \mathfrak{g}_2\}, \qquad \{\mathfrak{g}_1, \mathfrak{g}_3\} \qquad \text{and} \qquad \{\mathfrak{g}_1, \mathfrak{g}_4\},$$

but need to derive

$$\{\mathfrak{g}_2, \mathfrak{g}_3\}, \qquad \{\mathfrak{g}_2, \mathfrak{g}_4\}, \qquad \{\mathfrak{g}_3, \mathfrak{g}_4\}.$$

In fact, once we have determined the first of these, the remaining pair follow by the Jacobi identity.

We introduce the following notation for the basis elements of $\hat{\mathfrak{g}}$:

$$\mathfrak{g}_k = \{e_k, h_k, f_k\}, \qquad \text{for} \quad k = 1, 2, 3, \qquad \text{and} \qquad \mathfrak{g}_4 = \{h_4\},$$

with $\mathfrak{g}_1 + \mathfrak{g}_2$ satisfying the relations given by Table 1 (with $a = a_2$, possibly zero) and $\mathfrak{g}_1 + \mathfrak{g}_3$ satisfying the relations given by Table 1 (with $a = a_3$, possibly zero). We also have that h_4 commutes with $\mathfrak{g}_1$.

For $\hat{\mathfrak{g}}$ to be a Lie algebra, we must have

$$\{\mathfrak{g}_2, \mathfrak{g}_3\} \subset \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3 + \mathfrak{g}_4.$$

Noting that $\{\{e_2, h_3\}, h_1\} = 2\{e_2, h_3\}$, we have

$$\{e_2, h_3\} = \alpha e_1 + \beta e_2 + \gamma e_3, \qquad \text{for arbitrary constants } \alpha, \ \beta, \ \gamma.$$

We can repeat the argument of Proposition 3.1 to show that, without loss of generality, we may choose $\beta = \gamma = 0$. Defining

$$\hat{e}_2 = e_2 + \mu e_1 \quad \Rightarrow \quad \hat{h}_2 = h_2 - \mu h_1 \qquad \text{and} \qquad \hat{f}_2 = f_2 - 2\mu f_1,$$
$$\hat{e}_3 = e_3 + \nu e_1 \quad \Rightarrow \quad \hat{h}_3 = h_3 - \nu h_1 \qquad \text{and} \qquad \hat{f}_3 = f_3 - 2\nu f_1,$$

for arbitrary parameters μ, ν, then

$$\{\hat{e}_2, \hat{h}_3\} = (\alpha - 2\mu\nu)e_1 + (\beta - 2\nu)e_2 + (\gamma - 2\mu)e_3.$$

Choosing $\mu = \frac{1}{2}\gamma$, $\nu = \frac{1}{2}\beta$, we have

$$\{\hat{e}_2, \hat{h}_3\} = \hat{\alpha} e_1, \qquad \text{where} \qquad \hat{\alpha} = \alpha - \frac{1}{2}\beta\gamma.$$

Dropping "hats", we have shown that $\{e_2, h_3\} = a_4 e_1$, for some parameter a_4. The next proposition extends this to the whole of $\{\mathfrak{g}_2, \mathfrak{g}_3\}$, as shown in Table 2.

Proposition 4.1. *Bases can be chosen for $\mathfrak{g}_2$ and $\mathfrak{g}_3$, satisfying $\{\mathfrak{g}_2, \mathfrak{g}_3\} \subset \mathfrak{g}_1 + \mathfrak{g}_4$. Specifically, there exist parameters a_4, γ, such that the relations shown in Table 2 are satisfied.*

Table 2. The relations for $\{\mathfrak{g}_2, \mathfrak{g}_3\}$.

	e_3	h_3	f_3
e_2	0	$a_4 e_1$	$\gamma h_4 - a_4 h_1$
h_2	$-a_4 e_1$	$-\gamma h_4$	$-2a_4 f_1$
f_2	$\gamma h_4 + a_4 h_1$	$2a_4 f_1$	0

Proof. First, we note that since $\{\{e_2, e_3\}, h_1\} = 4\{e_2, e_3\}$, we have $\{e_2, e_3\} = 0$. Similarly, we find $\{f_2, f_3\} = 0$. We then have

$$\{\{e_2, e_3\}, f_1\} = 0 \quad \Rightarrow \quad \{h_2, e_3\} = -\{e_2, h_3\} = -a_4 e_1,$$
$$\{\{f_2, f_3\}, e_1\} = 0 \quad \Rightarrow \quad \{h_2, f_3\} + \{f_2, h_3\} = 0.$$

Further action of f_1 leads to

$$\{h_2, h_3\} + \{e_2, f_3\} = -a_4 h_1 \qquad \text{and} \qquad \{f_2, e_3\} + \{h_2, h_3\} = a_4 h_1.$$

Since $\{\{h_2, h_3\}, h_1\} = 0$, we have $\{h_2, h_3\} = -\gamma h_4 + \delta h_1$, so, bracketing this with f_1 gives

$$\{h_2, f_3\} + \{f_2, h_3\} = 2\delta f_1 \quad \Rightarrow \quad \delta = 0.$$

Piecing these results together, we obtain Table 2.   ■

Now that we have $\{\mathfrak{g}_2, \mathfrak{g}_3\}$, we calculate $\{\mathfrak{g}_2, \mathfrak{g}_4\}$ and $\{\mathfrak{g}_3, \mathfrak{g}_4\}$ by using the Jacobi identity. We require $\gamma \neq 0$ if $\mathfrak{g}_4$ is to enter our calculations, so, without loss of generality, we may take $\gamma = 1$, but leave a_4 arbitrary.

Since $\{h_2, h_3\} = -h_4$, we have

$$\{e_2, h_4\} = -\{e_2, \{h_2, h_3\}\} = \{h_2, \{h_3, e_2\}\} + \{h_3, \{e_2, h_2\}\} = 2(a_2 e_3 - a_4 e_2),$$

using the relations we already have. Similarly, we can derive the remaining brackets to complete Table 3. The lower part of the table is, of course, determined by skew symmetry.

Table 3. The 10-dimensional conformal algebra when $\{\mathfrak{g}_i, \mathfrak{g}_i\} \subset \mathfrak{g}_1$.

	e_1	h_1	f_1	e_2	h_2	f_2	e_3	h_3	f_3	h_4
e_1	0	$2e_1$	$-h_1$	0	$-2e_2$	$-2h_2$	0	$-2e_3$	$-2h_3$	0
h_1		0	$2f_1$	$-2e_2$	0	$2f_2$	$-2e_3$	0	$2f_3$	0
f_1			0	$-h_2$	$-f_2$	0	$-h_3$	$-f_3$	0	0
e_2				0	$a_2 e_1$	$-a_2 h_1$	0	$a_4 e_1$	$h_4 - a_4 h_1$	$2(a_2 e_3 - a_4 e_2)$
h_2					0	$-2a_2 f_1$	$-a_4 e_1$	$-h_4$	$-2a_4 f_1$	$2(a_2 h_3 - a_4 h_2)$
f_2						0	$h_4 + a_4 h_1$	$2a_4 f_1$	0	$2(a_2 f_3 - a_4 f_2)$
e_3							0	$a_3 e_1$	$-a_3 h_1$	$2(a_4 e_3 - a_3 e_2)$
h_3								0	$-2a_3 f_1$	$2(a_4 h_3 - a_3 h_2)$
f_3									0	$2(a_4 f_3 - a_3 f_2)$
h_4										0

The cases for which $\{\mathfrak{g}_2, \mathfrak{g}_2\} = \mathbf{0}$ and/or $\{\mathfrak{g}_3, \mathfrak{g}_3\} = \mathbf{0}$ are obtained by setting $a_2 = 0$ and/or $a_3 = 0$.

The Lie algebra automorphisms ι_1 and ι_2

The automorphisms of $\mathfrak{g}_1$, given by (2.6) also act on this 10-dimensional algebra:

	e_1	h_1	f_1	e_2	h_2	f_2	e_3	h_3	f_3	h_4
ι_1:	f_1	$-h_1$	e_1	$-\frac{1}{2}f_2$	$-h_2$	$-2e_2$	$-\frac{1}{2}f_3$	$-h_3$	$-2e_3$	h_4
ι_{23}:	$-e_1$	h_1	$-f_1$	e_2	$-h_2$	f_2	e_3	$-h_3$	f_3	h_4

Note that the four spaces $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$ and $\mathfrak{g}_4$ are each invariant.

4.2.1   The equations for the coefficients

Table 3 was obtained from Table 2 by requiring *algebraic consistency* as an abstract Poisson algebra. However, these Poisson relations impose additional *differential* relations on the functions used to define the basis elements. We will solve the resulting equations for (A_3, C_3) in terms of (A_2, C_2), which will be arbitrary solutions of equations (3.4) or (3.8).

From Case 3 of Section 3.2 or the Abelian case of Section 3.3, we have

$$e_i = A_i p_1 + C_i p_3,$$
$$h_i = 2A_i q_2 p_1 + 2(A_i q_1 - C_i q_3) p_2 + 2C_i q_2 p_3,$$
$$f_i = 2\big(A_i\big(q_1^2 + q_2^2 + q_3^2\big) - 2C_i q_1 q_3\big) p_1 + 4q_2(A_i q_1 - C_i q_3) p_2$$
$$\qquad + 2\big(C_i\big(q_2^2 - q_1^2 - q_3^2\big) + 2A_i q_1 q_3\big) p_3,$$

where $i = 2, 3$ and $A_i(r_0)$, $C_i(r_0)$ satisfy either (3.4) (with parameter $a \to a_i$) or (3.8), as well as

$$h_4 = C_4(r_0)(q_3 p_1 + q_1 p_3),$$

as the basis of $\mathfrak{g}_4$.

We must solve the two equations

$$\{e_2, h_3\} = a_4 e_1, \tag{4.1a}$$
$$\{h_2, h_3\} = -h_4, \tag{4.1b}$$

each of which has 3 components (the coefficients of p_i).

Equations (3.4) (or (3.8)), together with the p_3 component of (4.1a), can be used to eliminate the derivatives $A_i'(r_0)$ and $C_i'(r_0)$ (for $i = 2, 3$), and then the p_3 component of (4.1b) gives the formula

$$C_4 = 4(A_2 C_3 - A_3 C_2).$$

It is then possible to solve the remaining parts of (4.1) for $A_3(r_0)$ and $C_3(r_0)$, but the solution depends upon whether or not $a_2 a_3 = 0$.

4.3 The case $\{\mathfrak{g}_2, \mathfrak{g}_2\} \subset \mathfrak{g}_1$ and $\{\mathfrak{g}_3, \mathfrak{g}_3\} \subset \mathfrak{g}_1$

When $a_2 a_3 \neq 0$, we obtain

$$
\begin{aligned}
A_3(r_0) &= \frac{\sqrt{a_2 a_3 - a_4^2}(2r_0 A_2 C_2 - 2A_2^2 - a_2)}{a_2\sqrt{2\big(a_2\big(r_0^2 - 1\big) - 2(r_0 C_2 - A_2)^2\big)}} + \frac{a_4 A_2}{a_2}, \\
C_3(r_0) &= \frac{\sqrt{a_2 a_3 - a_4^2}(2r_0 C_2^2 - 2A_2 C_2 - a_2 r_0)}{a_2\sqrt{2\big(a_2\big(r_0^2 - 1\big) - 2(r_0 C_2 - A_2)^2\big)}} + \frac{a_4 C_2}{a_2}, \\
C_4(r_0) &= 4(A_2 C_3 - A_3 C_2) = \frac{4\sqrt{a_2 a_3 - a_4^2}(C_2 - r_0 A_2)}{\sqrt{2\big(a_2\big(r_0^2 - 1\big) - 2(r_0 C_2 - A_2)^2\big)}},
\end{aligned}
\tag{4.2}
$$

where $A_2(r_0)$, $C_2(r_0)$ are arbitrary solutions of equation (3.4) with $a = a_2$.

4.3.1 Casimir functions and conformal factors

Table 3 can be rearranged by re-ordering the 4 subspaces of $\hat{\mathfrak{g}}$. We can take

- $\mathfrak{g}_1 + \mathfrak{g}_2$ as Killing vectors of a Casimir H_{12}, with conformal Killing vectors in the space $\mathfrak{g}_3 + \mathfrak{g}_4$.

- $\mathfrak{g}_1 + \mathfrak{g}_3$ as Killing vectors of a Casimir H_{13}, with conformal Killing vectors in the space $\mathfrak{g}_2 + \mathfrak{g}_4$.

- $\mathfrak{g}_1 + \mathfrak{g}_4$ as Killing vectors of a Casimir H_{14}, with conformal Killing vectors in the space $\mathfrak{g}_2 + \mathfrak{g}_3$.

Since they all have the same 10-dimensional conformal algebra, they are conformally equivalent to one another.

The Hamiltonian H_{12} will denote the Casimir corresponding to the sub-algebra $\mathfrak{g}_1 + \mathfrak{g}_2$, and is given by (3.5), but with $(a, A, C) = (a_2, A_2, C_2)$. This corresponds to a metric of *constant curvature*, with $R = -12a_2$. The 6-dimensional algebra $\mathfrak{g}_1 + \mathfrak{g}_2$ is just the symmetry algebra and $\mathfrak{g}_3 + \mathfrak{g}_4$ correspond to conformal symmetries, satisfying

$$\{e_3, H_{12}\} = w_{31}H_{12}, \qquad \{h_3, H_{12}\} = w_{32}H_{12},$$
$$\{f_3, H_{12}\} = w_{33}H_{12}, \qquad \{h_4, H_{12}\} = w_{34}H_{12}.$$

We need to calculate w_{31} directly, obtaining

$$w_{31} = \frac{a_2 a_3 - a_4^2}{a_2(q_1 A_3 - q_3 C_3) - a_4(q_1 A_2 - q_3 C_2)},$$

but the remaining (infinitesimal) conformal factors can be derived by using the Poisson bracket relations of Table 3:

$$w_{32} = \{w_{31}, f_1\} = 2q_2 w_{31}, \qquad w_{33} = \{w_{32}, f_1\} = 2(q_2^2 + q_3^2 - q_1^2)w_{31},$$

and

$$\{h_4, H_{12}\} = \{\{h_3, h_2\}, H_{12}\} = \{\{h_3, H_{12}\}, h_2\} = \{w_{32}, h_2\}H_{12}$$
$$\Rightarrow \quad w_{34} = 4(q_1 A_2 - q_3 C_2)w_{31}.$$

Remark 4.2. The 3 functions w_{3i} form a representation space for our algebra $\mathfrak{g}_1$. Under the action of the Poisson bracket, we have

$$f_1: \ (w_{31}, w_{32}, w_{33}) \mapsto (w_{32}, w_{33}, 0), \qquad h_1: \ (w_{31}, w_{32}, w_{33}) \mapsto (2w_{31}, 0, -2w_{33}),$$
$$e_1: \ (w_{31}, w_{32}, w_{33}) \mapsto (0, 2w_{31}, 2w_{32}).$$

The function w_{34} is invariant with respect to $\mathfrak{g}_1$.

The Hamiltonian H_{13} corresponds to the sub-algebra $\mathfrak{g}_1 + \mathfrak{g}_3$ and is again of the form (3.5), but now with $(a, A, C) = (a_3, A_3, C_3)$, so corresponds to a metric of *constant curvature*, with $R = -12a_3$. The 6-dimensional algebra $\mathfrak{g}_1 + \mathfrak{g}_3$ is now the symmetry algebra and $\mathfrak{g}_2 + \mathfrak{g}_4$ correspond to conformal symmetries.

Since H_{13} has the same conformal algebra as H_{12}, the corresponding metrics must be conformally related. To see this (on the level of the inverse metric) we use formulae (4.2) to replace A_3, C_3 in H_{13} to obtain

$$H_{13} = \phi_{13}H_{12}, \quad \text{where} \quad \phi_{13} = \phi_{13}^0 + \frac{2a_4\sqrt{2(a_2 a_3 - a_4^2)}}{a_2^2}\phi_{13}^1 + \frac{a_4^2}{a_2^2}\phi_{13}^2,$$

where

$$\phi_{13}^0 = \frac{2a_3(q_1 A_2 - q_3 C_2)^2}{a_2(a_2(q_3^2 - q_1^2) - 2(q_1 A_2 - q_3 C_2)^2)},$$

$$\phi_{13}^1 = \frac{q_3 C_2 - q_1 A_2}{\sqrt{a_2(a_2(q_3^2 - q_1^2) - 2(q_1 A_2 - q_3 C_2)^2)}},$$

$$\phi_{13}^2 = \frac{a_2(q_3^2 - q_1^2) - 4(q_1 A_2 - q_3 C_2)^2}{a_2(q_3^2 - q_1^2) - 2(q_1 A_2 - q_3 C_2)^2}.$$

Since $\{e_2, H_{12}\} = 0$, we have

$$\{e_2, H_{13}\} = \{e_2, \log(\phi_{13})\}H_{13},$$

giving

$$w_{21} = \{e_2, \log(\phi_{13})\}$$

$$= \frac{a_2\sqrt{2(a_2 a_3 - a_4^2)}}{\sqrt{2(a_2 a_3 - a_4^2)}(q_1 A_2 - q_3 C_2) - a_4\sqrt{a_2(q_3^2 - q_1^2) - 2(q_1 A_2 - q_3 C_2)^2}}.$$

Again, with the notation $\{h_2, H_{13}\} = w_{22}H_{13}$, $\{f_2, H_{13}\} = w_{23}H_{13}$, we use the action of f_1 to find

$$w_{22} = 2q_2 w_{21}, \qquad w_{23} = 2(q_2^2 + q_3^2 - q_1^2)w_{21}.$$

We can then use $h_4 = \{h_3, h_2\}$ to obtain

$$w_{24} = -\frac{2}{a_2}\left(2a_4(q_1 A_2 - q_3 C_2) + \sqrt{2(a_2 a_3 - a_4^2)}\sqrt{a_2(q_3^2 - q_1^2) - 2(q_1 A_2 - q_3 C_2)^2}\right)w_{21},$$

where $\{h_4, H_{13}\} = w_{24}H_{13}$.

The Hamiltonian H_{14} corresponds to the sub-algebra $\mathfrak{g}_1 + \mathfrak{g}_4$. Since $\mathfrak{g}_4$ contains the single element h_4, defined by Z_{41} of (2.10), with C_4 given by (4.2), and since $\{\mathfrak{g}_1, \mathfrak{g}_4\} = \{\mathfrak{g}_4, \mathfrak{g}_4\} = \mathbf{0}$, this is an algebraically trivial extension, since it is just a *direct sum*. However, the Casimir,

$$H_{14} = e_1 f_1 + \frac{1}{4}h_1^2 + \alpha h_4^2 = (r_0^2 + \alpha C_4^2(r_0))r_1^2 + (r_0^2 - 1)r_2$$

$$= (q_1^2 + \alpha q_3^2 C_4^2)p_1^2 + 2q_1 q_3(1 + \alpha C_4^2)p_1 p_3 + (q_3^2 - q_1^2)p_2^2 + (q_3^2 + \alpha q_1^2 C_4^2)p_3^2,$$

defines a non-degenerate upper-index metric whenever $\alpha \neq 0$, which is conformally equivalent to H_{12} when $\alpha = \frac{1}{4(a_2 a_3 - a_4^2)}$, satisfying

$$H_{14} = \phi_{14}H_{12}, \qquad \text{where} \qquad \phi_{14} = \frac{q_3^2 - q_1^2}{2a_2(q_3^2 - q_1^2) - 4(q_1 A_2 - q_3 C_2)^2}.$$

The metric, corresponding to H_{14}, has constant *scalar* curvature $R = -2$, but is not actually a constant curvature metric, since it does not satisfy (3.6) and, indeed, only has a 4-dimensional symmetry algebra.

The elements of $\mathfrak{g}_2 + \mathfrak{g}_3$ correspond to conformal symmetries of H_{14}. Defining z_{ki} by

$$\{e_k, H_{14}\} = z_{k1}H_{14}, \qquad \{h_k, H_{14}\} = z_{k2}H_{14}, \qquad \{f_k, H_{14}\} = z_{k3}H_{14}, \qquad \text{for} \quad k = 2, 3,$$

we again have $z_{21} = \{e_2, \log(\phi_{14})\}$ and use the action of f_1 to find

$$z_{21} = \frac{2(q_1 A_2 - q_3 C_2)}{q_3^2 - q_1^2}, \qquad z_{22} = 2q_2 z_{21}, \qquad z_{23} = 2(q_2^2 + q_3^2 - q_1^2)z_{21}.$$

Noting that

$$\{e_3, H_{14}\} = \{e_3, \log(\phi_{14})\}H_{14} + \phi_{14}\{e_3, H_{12}\} \quad \Rightarrow \quad z_{31} = \{e_3, \log(\phi_{14})\} + w_{31},$$

we find that z_{3i} are given by the same formulae as z_{2i}, but with (A_2, C_2) replaced by (A_3, C_3).

Remark 4.3. The function ϕ_{14} satisfies

$$\phi_{14} = \frac{1}{8(a_2 a_3 - a_4^2)}\left(w_{31}w_{33} - \frac{1}{2}w_{32}^2\right),$$

which is an invariant of the representation mentioned in Remark 4.2.

The Lie algebra automorphisms ι_1 and ι_2

Under the action of ι_1 of (2.6a), we have

$$\frac{q_3}{q_1} \mapsto \frac{q_3}{q_1}, \qquad q_1^2 - q_2^2 - q_3^2 \mapsto \frac{1}{q_1^2 - q_2^2 - q_3^2}, \qquad (H_{12}, H_{13}, H_{14}) \mapsto (H_{12}, H_{13}, H_{14}).$$

For $k = 2, 3$, the functions

$$(w_{k1}, w_{k2}, w_{k3}, w_{k4}) \mapsto \left(-\frac{1}{2} w_{k3}, -w_{k2}, -2 w_{k1}, w_{k4} \right) \qquad \text{and} \qquad (\phi_{13}, \phi_{14}) \mapsto (\phi_{13}, \phi_{14}),$$

and similarly for z_{ki}. The action of ι_2 is even simpler.

These automorphisms will be very important in later sections, when we discuss superintegrable systems associated with some of our Casimir functions.

4.3.2 Reduction to the diagonal case

Consider the Hamiltonian H_{12}, which is of the form (3.5), but with $(a, A, C) = (a_2, A_2, C_2)$. The only off-diagonal term is the coefficient of $p_1 p_3$, which vanishes when $2A_2^2 - 2C_2^2 + a_2 = 0$, which then implies that the right-hand side of (3.4) also vanishes, so we have

$$A_2' - r_0 C_2' = 0, \qquad A_2 A_2' - C_2 C_2' = 0 \quad \Rightarrow \quad (r_0 A_2 - C_2) C_2' = 0.$$

Since we require that $r_0 A_2 - C_2 \neq 0$, we have

$$C_2 = c_1 \quad \text{(a constant)} \quad \Rightarrow \quad A_2 = \sqrt{\frac{2c_1^2 - a_2}{2}},$$

so

$$H_{12} = 2\left(\sqrt{2} c_1 q_1 - \sqrt{2c_1^2 - a_2} q_3 \right)^2 \left(p_1^2 - p_2^2 - p_3^2 \right) = 2 a_2 q_1^2 \left(p_1^2 - p_2^2 - p_3^2 \right),$$

when c_1 is chosen so that $2c_1^2 = a_2$. This is just the case of equations (3.16). For the choice $a_2 = -a_3 = 2, a_4 = 0$, the conformal algebra has the explicit form:

$$
\begin{aligned}
e_1 &= p_2, & h_1 &= -2(q_1 p_1 + q_2 p_2 + q_3 p_3), \\
f_1 &= -2 q_1 q_2 p_1 + (q_3^2 - q_1^2 - q_2^2) p_2 - 2 q_2 q_3 p_3, \\
e_2 &= p_3, & h_2 &= 2(q_2 p_3 - q_3 p_2), & f_2 &= -4 q_3 (q_1 p_1 + q_2 p_2) - 2(q_1^2 - q_2^2 + q_3^2) p_3, \quad (4.3) \\
e_3 &= p_1, & h_3 &= 2(q_1 p_2 + q_2 p_1), & f_3 &= 2(q_1^2 + q_2^2 + q_3^2) p_1 + 4 q_1 (q_2 p_2 + q_3 p_3), \\
h_4 &= -4(q_3 p_1 + q_1 p_3),
\end{aligned}
$$

which is a 10-dimensional extension of the algebra (3.16c). In this case we have

$$
\begin{aligned}
H_{12} &= 4 q_1^2 (p_1^2 - p_2^2 - p_3^2), & H_{13} &= 4 q_3^2 (p_1^2 - p_2^2 - p_3^2), \\
H_{14} &= (q_1^2 - q_3^2)(p_1^2 - p_2^2 - p_3^2).
\end{aligned}
\quad (4.4)
$$

Remark 4.4 (further automorphism)**.** As can be seen, the Casimir H_{12} is invariant under the interchange $2 \leftrightarrow 3$, which induces the following involution ι_{23} of the 10-dimensional algebra

$$(e_1, h_1, f_1, e_2, h_2, f_2, e_3, h_3, f_3, h_4) \mapsto \left(e_2, h_1, \frac{1}{2} f_2, e_1, -h_2, 2 f_1, e_3, -\frac{1}{2} h_4, f_3, -2 h_3 \right). \quad (4.5)$$

This is no longer an automorphism of $\mathfrak{g}_1$, so its representation spaces are not individually preserved, but it is an automorphism of the symmetry algebra $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ and of the conformal elements $\mathfrak{g}_3 + \mathfrak{g}_4$.

4.3.3 H_{12} of (4.4) as a reduction from flat space in 4-dimensions

Starting with $\mathfrak{g}_1$ of (4.3), we can build a 3-dimensional, highest weight representation in the space of functions of q_1, q_2, q_3. We obtain

$$y_1 = \frac{1}{q_1}, \qquad y_2 = \frac{2q_2}{q_1}, \qquad y_3 = \frac{2(q_2^2 + q_3^2 - q_1^2)}{q_1},$$

satisfying $\{(y_1, y_2, y_3), e_1\} = (0, 2y_1, 2y_2)$, $\{(y_1, y_2, y_3), h_1\} = (2y_1, 0, -2y_3)$, $\{(y_1, y_2, y_3), f_1\} = (y_2, y_3, 0)$. When acting on these with $\mathfrak{g}_2$, we need to add the function $y_4 = \frac{2q_3}{q_1}$, which Poisson commutes with the whole of $\mathfrak{g}_1$ (it is just $2r_0$ (see (2.3))). The action of $\mathfrak{g}_2$ is given by

$$\{(y_1, y_2, y_3, y_4), e_2\} = (0, 0, 2y_4, 2y_1), \qquad \{(y_1, y_2, y_3, y_4), h_2\} = (0, -2y_4, 0, 2y_2),$$
$$\{(y_1, y_2, y_3, y_4), f_2\} = (2y_4, 0, 0, 2y_3).$$

These clearly define a linear action of $\mathfrak{g}_1 + \mathfrak{g}_2$ on $\{y_i\}_{i=1}^4$, given by

$$T_x f = \{f, x\}, \qquad \text{satisfying} \qquad [T_x, T_y] f = -T_{\{x,y\}} f.$$

The four variables y_i satisfy the quadratic constraint $2y_1 y_3 - y_2^2 - y_4^2 = -4$, which defines a quadratic form with matrix

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and the matrices T_x for $x \in \mathfrak{g}_1 + \mathfrak{g}_2$ are "infinitesimally orthogonal" with respect to this "metric", satisfying $T_x S + S T_x^t = 0$. This means that our symmetry algebra $\mathfrak{g}_1 + \mathfrak{g}_2$ is just $\mathfrak{so}(1,3)$. If we use the matrix S to define the corresponding Lorentzian metric, we find

$$\mathrm{d}s^2 = 2\mathrm{d}y_1 \mathrm{d}y_3 - \mathrm{d}y_2^2 - \mathrm{d}y_4^2 = \frac{4}{q_1^2}\big(\mathrm{d}q_1^2 - \mathrm{d}q_2^2 - \mathrm{d}q_3^2\big),$$

corresponding to the Hamiltonian H_{12} of (4.4).

4.4   The case $\{\mathfrak{g}_2, \mathfrak{g}_2\} = 0$ and $\{\mathfrak{g}_3, \mathfrak{g}_3\} \subset \mathfrak{g}_1$

Here we must solve equations (4.1) with $a_2 = 0$, so $A_2(r_0)$, $C_2(r_0)$ satisfy equation (3.8), while $A_3(r_0)$, $C_3(r_0)$ satisfy equation (3.4), with $a = a_3$, which can be either zero or non-zero. The calculation soon gives the choice of

- $a_4 = 0$, leading to $A_2 = r_0 C_2$, which means that the determinant condition (3.10) is not satisfied, so the Casimir function (3.7) cannot be associated with a flat metric.

- $a_4 \neq 0$, which leads to a non-degenerate flat metric, but has a *singular limit* as $a_4 \to 0$. This is the only case we consider here.

When $a_4 \neq 0$ we find

$$A_3 = \frac{a_3 A_2}{2a_4} + \frac{a_4\big(2q_1 q_3 C_2 - (q_1^2 + q_3^2)A_2\big)}{4(q_1 A_2 - q_3 C_2)^2}, \qquad C_3 = \frac{a_3 C_2}{2a_4} + \frac{a_4\big((q_1^2 + q_3^2)C_2 - 2q_1 q_3 A_2\big)}{4(q_1 A_2 - q_3 C_2)^2},$$
$$C_4 = 4(A_2 C_3 - A_3 C_2) = \frac{2a_4(q_1 C_2 - q_3 A_2)}{q_1 A_2 - q_3 C_2},$$

where $A_2(r_0)$, $C_2(r_0)$ are arbitrary solutions of equation (3.8).

4.4.1 Casimir functions and conformal factors

We now have Table 3, with $a_2 = 0$ and again consider various 6-dimensional subalgebras and their respective Casimir functions. The automorphisms (2.6) are still valid in this case.

The Hamiltonian H_{12} will again denote the Casimir corresponding to the sub-algebra $\mathfrak{g}_1 + \mathfrak{g}_2$, and is given by (3.9), but with $(A, C) = (A_2, C_2)$. This corresponds to a *flat* metric. The 6-dimensional algebra $\mathfrak{g}_1 + \mathfrak{g}_2$ is just the symmetry algebra and $\mathfrak{g}_3 + \mathfrak{g}_4$ correspond to conformal symmetries, satisfying

$$\{e_3, H_{12}\} = w_{31}H_{12}, \qquad \{h_3, H_{12}\} = w_{32}H_{12},$$
$$\{f_3, H_{12}\} = w_{33}H_{12}, \qquad \{h_4, H_{12}\} = w_{34}H_{12}.$$

The coefficients are calculated in the same way to give

$$w_{31} = \frac{a_4}{q_1 A_2 - q_3 C_2}, \qquad w_{32} = 2q_2 w_{31}, \qquad w_{33} = 2(q_2^2 + q_3^2 - q_1^2)w_{31}, \qquad w_{34} = 4a_4.$$

The Hamiltonian H_{13} corresponds to the sub-algebra $\mathfrak{g}_1 + \mathfrak{g}_3$ and is of the form (3.5), with $(a, A, C) = (a_3, A_3, C_3)$, so corresponds to a metric of *constant curvature*, with $R = -12a_3$. The 6-dimensional algebra $\mathfrak{g}_1 + \mathfrak{g}_3$ is now the symmetry algebra and $\mathfrak{g}_2 + \mathfrak{g}_4$ correspond to conformal symmetries.

Again H_{13} is conformally related to H_{12}, with

$$H_{13} = \phi_{13}H_{12}, \qquad \text{where} \qquad \phi_{13} = \frac{\left(2a_3(q_1 A_2 - q_3 C_2)^2 - a_4^2(q_3^2 - q_1^2)\right)^2}{16a_4^2(q_1 A_2 - q_3 C_2)^4}$$

Defining w_{2k} by

$$\{e_2, H_{13}\} = w_{21}H_{13}, \qquad \{h_2, H_{13}\} = w_{22}H_{13},$$
$$\{f_2, H_{13}\} = w_{23}H_{13}, \qquad \{h_4, H_{13}\} = w_{24}H_{13},$$

we have

$$w_{21} = \frac{4a_4^2(q_1 A_2 - q_3 C_2)}{a_4^2(q_3^2 - q_1^2) - 2a_3(q_1 A_2 - q_3 C_2)^2}, \qquad w_{22} = 2q_2 w_{21},$$
$$w_{23} = 2(q_2^2 + q_3^2 - q_1^2)w_{21}, \qquad w_{24} = -\frac{a_4^2(q_3^2 - q_1^2) + 2a_3(q_1 A_2 - q_3 C_2)^2}{a_4(q_1 A_2 - q_3 C_2)}w_{21}.$$

The Hamiltonian H_{14} corresponds to the sub-algebra $\mathfrak{g}_1 + \mathfrak{g}_4$, and is given by

$$H_{14} = e_1 f_1 + \frac{1}{4}h_1^2 + \alpha h_4^2 = (r_0^2 + \alpha C_4^2(r_0))r_1^2 + (r_0^2 - 1)r_2$$
$$= (q_1^2 + \alpha q_3^2 C_4^2)p_1^2 + 2q_1 q_3(1 + \alpha C_4^2)p_1 p_3 + (q_3^2 - q_1^2)p_2^2 + (q_3^2 + \alpha q_1^2 C_4^2)p_3^2,$$

which is non-degenerate whenever $\alpha \neq 0$, and is conformally equivalent to H_{12} when $\alpha = \frac{-1}{4a_4^2}$, satisfying

$$H_{14} = \phi_{14}H_{12}, \qquad \text{where} \qquad \phi_{14} = -\frac{q_3^2 - q_1^2}{4(q_1 A_2 - q_3 C_2)^2}.$$

As before, the metric, corresponding to H_{14}, has constant *scalar* curvature $R = -2$, but is not actually a constant curvature metric, since it does not satisfy (3.6) and, indeed, only has a 4-dimensional symmetry algebra.

The elements of $\mathfrak{g}_2 + \mathfrak{g}_3$ correspond to conformal symmetries of H_{14}. Defining z_{ki} by

$$\{e_k, H_{14}\} = z_{k1}H_{14}, \qquad \{h_k, H_{14}\} = z_{k2}H_{14}, \qquad \{f_k, H_{14}\} = z_{k3}H_{14}, \qquad \text{for} \quad k = 2, 3,$$

we use the action of f_1 to find

$$z_{21} = \frac{2(q_1 A_2 - q_3 C_2)}{q_3^2 - q_1^2}, \qquad z_{22} = 2q_2 z_{21}, \qquad z_{23} = 2(q_2^2 + q_3^2 - q_1^2)z_{21},$$

with z_{3i} being given by the same formulae as z_{2i}, but with (A_2, C_2) replaced by (A_3, C_3).

4.4.2 Reduction to the diagonal case

The diagonalisation of the Hamiltonian H_{12} is simpler in the flat case. The only off-diagonal term is the coefficient of $p_1 p_3$, which now vanishes when $A_2^2 - C_2^2 = 0$, so $C_2 = \pm A_2$, corresponding to

$$H_{12} = 4(q_1 \mp q_3)^2 A_2^2 (p_1^2 - p_2^2 - p_3^2).$$

For simplicity, we choose $C_2 = A_2 = \frac{1}{2}$, after which we find

$$H_{12} = (q_1 - q_3)^2 (p_1^2 - p_2^2 - p_3^2), \qquad H_{13} = \left(a_4(q_1 + q_3) + \frac{a_3}{2a_4}(q_1 - q_3)\right)^2 (p_1^2 - p_2^2 - p_3^2),$$

$$H_{14} = (q_1^2 - q_3^2)(p_1^2 - p_2^2 - p_3^2). \tag{4.6}$$

The conformal algebra now has the explicit form:

$$e_1 = p_2, \quad h_1 = -2(q_1 p_1 + q_2 p_2 + q_3 p_3), \qquad f_1 = -2q_1 q_2 p_1 + (q_3^2 - q_1^2 - q_2^2)p_2 - 2q_2 q_3 p_3,$$

$$e_2 = \frac{1}{2}(p_1 + p_3), \qquad h_2 = q_2(p_1 + p_3) + (q_1 - q_3)p_2,$$

$$f_2 = (q_2^2 + (q_1 - q_3)^2)p_1 + 2q_2(q_1 - q_3)p_2 + (q_2^2 - (q_1 - q_3)^2)p_3, \tag{4.7}$$

$$e_3 = -\frac{1}{2}a_4(p_1 - p_3) + \frac{a_3}{2a_4}e_2, \qquad h_3 = -a_4(q_2(p_1 - p_3) + (q_1 + q_3)p_2) + \frac{a_3}{2a_4}h_2,$$

$$f_3 = -a_4\big(((q_1 + q_3)^2 + q_2^2)p_1 + 2q_2(q_1 + q_3)p_2 + ((q_1 + q_3)^2 - q_2^2)p_3\big) + \frac{a_3}{2a_4}f_2,$$

$$h_4 = 2a_4(q_3 p_1 + q_1 p_3).$$

This is an extension of the algebra given in (3.17).

Remark 4.5. As can be seen, this algebra is no longer invariant under the involution ι_{23}, given by (4.5).

5 Adding potentials: separability

In this section we consider Hamiltonian systems of the form

$$H = H_0 + h(\boldsymbol{q}),$$

with the kinetic energy H_0 being one of our diagonal cases of H_{12}, given by (4.4) or (4.6), I_2 (of (2.5)) or H_{14}, given by (4.4).

For complete integrability (in the Liouville sense) we need two functions F_1 and F_2, such that H, F_1, F_2 are in involution:

$$\{H, F_1\} = \{H, F_2\} = \{F_1, F_2\} = 0. \tag{5.1}$$

We restrict attention to functions F_i, whose dependence on momenta is at most quadratic. Such functions will be the sum of two homogeneous parts, $F_i = F_i^{(2)} + F_i^{(0)}$, and

$$\{H, F_i\} = 0 \quad \Rightarrow \quad \{H_0, F_i^{(2)}\} = 0 \qquad \text{and} \qquad \{H_0, F_i^{(0)}\} + \{h, F_i^{(2)}\} = 0.$$

The first of these means that the coefficients of $p_i p_j$ in $F_i^{(2)}$ define a second order Killing tensor of the metric corresponding to H_0. When this metric is constant curvature, *all* Killing tensors are built as tensor products of Killing vectors (see [10]). In the Poisson representation, this just means that $F_i^{(2)}$ is some quadratic form of the elements of $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ (of Section 4.3.2 or Section 4.4.2). Since, in each case, this algebra is of rank 2, any $K \in \mathfrak{g}$ will commute with exactly one other element $\bar{K}$. Since we require $\{F_1^{(2)}, F_2^{(2)}\} = 0$, we must choose these quadratic parts to be independent quadratic form of some pair K, $\bar{K}$. For simplicity, we will choose our pairs to be one of e_1, e_2, or h_1, h_2 or f_1, f_2. In the conformally flat case of $H_0 = I_2$, we have a smaller symmetry algebra, but since I_2 is no longer the Casimir function, we can use H_0, $\mathcal{C}_1$ and a choice of K^2 to generate our integrals.

The choice of quadratic integrals means that our systems will be separable. The calculation of separable potentials is standard and it is well known that in the standard orthogonal coordinate systems, with separable kinetic energies, we can add potentials which depend upon a number of arbitrary functions of a single variable [12]. If a complete (possessing n parameters) solution of the Hamilton–Jacobi equation is found, then, by Jacobi's theorem, these parameters, when written in terms of the canonical variables, are quadratic (in momenta) first integrals of H. The problem has also been posed in the "opposite" direction: given a pair of Poisson commuting, homogeneously quadratic integrals (in two degrees of freedom) what sort of potentials can be added, whilst maintaining commutativity? This is a classical problem (see Whittaker [16, Chapter 12, Section 152]) and leads to the Bertrand–Darboux equation for the potential [13, 15]. This approach will be used in this section. The calculations are very similar, so the details will be omitted (with a few more included in the first case).

5.1 The constant curvature case of (4.4)

Here we consider the Hamiltonian

$$H = H_0 + h(\boldsymbol{q}), \qquad \text{where} \qquad H_0 = q_1^2(p_1^2 - p_2^2 - p_3^2),$$

with the kinetic energy $H_0 = \frac{1}{4}H_{12}$ of (4.4) and the specific conformal algebra (4.3).

5.1.1 The commuting pair h_1, h_2

Consider the case of

$$F_1 = \frac{1}{4}h_1^2 + g_1(\boldsymbol{q}) = (q_1 p_1 + q_2 p_2 + q_3 p_3)^2 + g_1(\boldsymbol{q}),$$
$$F_2 = \frac{1}{4}h_2^2 + g_2(\boldsymbol{q}) = (q_2 p_3 - q_3 p_2)^2 + g_2(\boldsymbol{q}).$$

Each of the equations (5.1) is *linear* in momenta, so give us 9 equations in all. This is an overdetermined system for the 3 functions h, g_1, g_2, which can be solved explicitly in terms of 3 functions, each of a single variable. We find

$$\partial_1 g_2 = 0, \qquad (q_2 \partial_2 + q_3 \partial_3)g_2 = 0 \quad \Rightarrow \quad g_2 = \varphi_2\left(\frac{q_3}{q_2}\right),$$

after which

$$q_2(q_2\partial_2 + q_3\partial_3)g_1 = (q_1\partial_1 + q_2\partial_2 + q_3\partial_3)g_2 = 0 \quad \Rightarrow \quad g_1 = \Phi_1(q_1, q_2^2 + q_3^2),$$

$$q_3(q_2\partial_2 + q_3\partial_3)h = q_1^2\partial_2 g_2 \quad \Rightarrow \quad h = \psi(q_1, q_2^2 + q_3^2) - \frac{q_1^2\varphi_2\left(\frac{q_3}{q_2}\right)}{q_2^2 + q_3^2}.$$

There are 2 more independent equations, leading to

$$(q_2\partial_1 + q_1\partial_2)\Phi_1 = 0 \quad \Rightarrow \quad \Phi_1(q_1, q_2^2 + q_3^2) = \varphi_1(q_1^2 - q_2^2 - q_3^2),$$

$$(q_1\partial_1 + q_2\partial_2 + q_3\partial_3)h = q_1\partial_1\Phi_1$$

$$\Rightarrow \quad \psi(q_1, q_2^2 + q_3^2) = \frac{q_1^2\varphi_1(q_1^2 - q_2^2 - q_3^2)}{q_1^2 - q_2^2 - q_3^2} + \varphi_3\left(\frac{q_2^2 + q_3^2}{q_1^2}\right).$$

In summary, we have

$$h = \frac{q_1^2\varphi_1(q_1^2 - q_2^2 - q_3^2)}{q_1^2 - q_2^2 - q_3^2} - \frac{q_1^2\varphi_2\left(\frac{q_3}{q_2}\right)}{q_2^2 + q_3^2} + \varphi_3\left(\frac{q_2^2 + q_3^2}{q_1^2}\right),$$

$$g_1 = \varphi_1(q_1^2 - q_2^2 - q_3^2), \qquad g_2 = \varphi_2\left(\frac{q_3}{q_2}\right).$$

This solution immediately gives us the separation variables

$$u = q_1^2 - q_2^2 - q_3^2, \qquad v = \frac{q_3}{q_2}, \qquad w = \frac{q_2^2 + q_3^2}{q_1^2}$$

$$\Rightarrow \quad h = \frac{\varphi_1(u)}{1 - w} - \frac{\varphi_2(v)}{w} + \varphi_3(w), \qquad g_1 = \varphi_1(u), \qquad g_2 = \varphi_2(v).$$

Remark 5.1 (action of automorphism)**.** Under the action of the automorphism ι_1 of (2.6a), we have

$$(u, v, w) \mapsto \left(\frac{1}{u}, -v, w\right),$$

so this solution is invariant up to redefining some arbitrary functions.

5.1.2 The commuting pairs e_1, e_2 and f_1, f_2

These two cases are connected by the action of the automorphism ι_1, of (2.6a). The simplest case to calculate is with the pair e_1, e_2:

$$F_1 = e_1^2 + g_1(\boldsymbol{q}) = p_2^2 + g_1(\boldsymbol{q}), \qquad F_2 = e_2^2 + g_2(\boldsymbol{q}) = p_3^2 + g_2(\boldsymbol{q}). \tag{5.2a}$$

The simple form of e_1 and e_2 means that we are already in separation coordinates, leading to

$$h = -q_1^2(\varphi_1(q_2) + \varphi_2(q_3)) + \varphi_3(q_1), \qquad g_1 = \varphi_1(q_2), \qquad g_2 = \varphi_2(q_3). \tag{5.2b}$$

The much more difficult case to calculate, involving f_1 and f_2, is simply obtained by using the automorphism ι_1, which preserves H_0, whilst mapping $e_1^2 \mapsto f_1^2$ and $e_2^2 \mapsto \frac{1}{4}f_2^2$. This gives

$$F_1 = f_1^2 + g_3(\boldsymbol{q}) = \left(-2q_1q_2p_1 + (q_3^2 - q_1^2 - q_2^2)p_2 - 2q_2q_3p_3\right)^2 + g_3(\boldsymbol{q}),$$

$$F_2 = \frac{1}{4}f_2^2 + g_4(\boldsymbol{q}) = \left(2q_3(q_1p_1 + q_2p_2) + (q_1^2 - q_2^2 + q_3^2)p_3\right)^2 + g_4(\boldsymbol{q}),$$

where

$$g_3 = \varphi_1\left(\frac{-q_2}{q_1^2 - q_2^2 - q_3^2}\right), \qquad g_4 = \varphi_2\left(\frac{q_3}{q_1^2 - q_2^2 - q_3^2}\right),$$

$$h = \varphi_3\left(\frac{q_1}{q_1^2 - q_2^2 - q_3^2}\right) - \frac{q_1^2\left(\varphi_1\left(\frac{-q_2}{q_1^2 - q_2^2 - q_3^2}\right) + \varphi_2\left(\frac{q_3}{q_1^2 - q_2^2 - q_3^2}\right)\right)}{\left(q_1^2 - q_2^2 - q_3^2\right)^2}.$$

5.2 The flat case of (4.6)

Here we consider the Hamiltonian

$$H = H_0 + h(\boldsymbol{q}), \qquad \text{where} \qquad H_0 = (q_1 - q_3)^2 (p_1^2 - p_2^2 - p_3^2), \tag{5.3}$$

with the kinetic energy $H_0 = H_{12}$ of (4.6) and the specific conformal algebra (4.7).

5.2.1 The commuting pair h_1, h_2

Consider the case of

$$F_1 = \frac{1}{4} h_1^2 + g_1(\boldsymbol{q}) = (q_1 p_1 + q_2 p_2 + q_3 p_3)^2 + g_1(\boldsymbol{q}),$$

$$F_2 = h_2^2 + g_2(\mathbf{q}) = (q_2(p_1 + p_3) + (q_1 - q_3)p_2)^2 + g_2(\mathbf{q}).$$

The relations $\{H, F_1\} = \{H, F_2\} = \{F_1, F_2\} = 0$ lead to

$$h = \frac{(q_1 - q_3)^2 \varphi_1 (q_1^2 - q_2^2 - q_3^2)}{q_1^2 - q_2^2 - q_3^2} - \varphi_2 \left(\frac{q_1 - q_3}{q_2} \right) + \varphi_3 \left(\frac{2q_1^2 - 2q_1 q_3 - q_2^2}{(q_1 - q_3)^2} \right),$$

$$g_1 = \varphi_1 (q_1^2 - q_2^2 - q_3^2), \qquad g_2 = \varphi_2 \left(\frac{q_1 - q_3}{q_2} \right).$$

Again, we have the separation variables

$$u = q_1^2 - q_2^2 - q_3^2, \qquad v = \frac{q_1 - q_3}{q_2}, \qquad w = \frac{2q_1^2 - 2q_1 q_3 - q_2^2}{(q_1 - q_3)^2},$$

with

$$h = \frac{\varphi_1(u)}{w - 1} - \varphi_2(v) + \varphi_3(w), \qquad g_1 = \varphi_1(u), \qquad g_2 = \varphi_2(v),$$

and again we have $(u, v, w) \mapsto \left(\frac{1}{u}, -v, w \right)$, under the automorphism ι_1, so the solution is invariant up to redefining some arbitrary functions.

5.2.2 The commuting pairs e_1, e_2 and f_1, f_2

These two cases are again connected by the action of the automorphism ι_1. The simplest case to calculate is with the pair e_1, e_2:

$$F_1 = e_1^2 + g_1(\mathbf{q}) = p_2^2 + g_1(\mathbf{q}), \qquad F_2 = 4e_2^2 + g_2(\mathbf{q}) = (p_1 + p_3)^2 + g_2(\mathbf{q}).$$

The relations $\{H, F_1\} = \{H, F_2\} = \{F_1, F_2\} = 0$ lead to

$$h = -(q_1 - q_3)^2 \varphi_1(q_2) + q_1 (q_1 - q_3)^2 \varphi_2'(q_1 - q_3) + \varphi_3(q_1 - q_3),$$

$$g_1 = \varphi_1(q_2), \qquad g_2 = \varphi_2(q_1 - q_3).$$

Again, the case involving f_1 and f_2 is simply obtained by using the automorphism ι_1, which preserves H_0, and still maps $e_1^2 \mapsto f_1^2$ and $e_2^2 \mapsto \frac{1}{4} f_2^2$. This gives

$$F_1 = f_1^2 + g_3(\mathbf{q}) = \left(-2q_1 q_2 p_1 + (q_3^2 - q_1^2 - q_2^2)p_2 - 2q_2 q_3 p_3 \right)^2 + g_3(\mathbf{q}),$$

$$F_2 = f_2^2 + g_4(\mathbf{q}) = \left((q_2^2 + (q_1 - q_3)^2)p_1 + 2q_2(q_1 - q_3)p_2 + (q_2^2 - (q_1 - q_3)^2)p_3 \right)^2 + g_4(\mathbf{q}),$$

where

$$g_3 = \varphi_1 \left(\frac{q_2}{q_1^2 - q_2^2 - q_3^2} \right), \qquad g_4 = \varphi_2 \left(\frac{q_1 - q_3}{q_1^2 - q_2^2 - q_3^2} \right),$$

$$h = -\frac{(q_1 - q_3)^2 \varphi_1 \left(\frac{q_2}{q_1^2 - q_2^2 - q_3^2} \right)}{(q_1^2 - q_2^2 - q_3^2)^2} + \frac{q_1 (q_1 - q_3)^2 \varphi_2' \left(\frac{q_1 - q_3}{q_1^2 - q_2^2 - q_3^2} \right)}{(q_1^2 - q_2^2 - q_3^2)^3} + \varphi_3 \left(\frac{q_1 - q_3}{q_1^2 - q_2^2 - q_3^2} \right).$$

5.3   Conformally flat cases

In this section we consider H_0 to be the diagonal case of I_2 (see (2.5))

$$H = H_0 + h(\boldsymbol{q}), \qquad \text{where} \qquad H_0 = \varphi\left(\frac{q_3}{q_1}\right) q_1^2 \big(p_1^2 - p_2^2 - p_3^2\big). \tag{5.4}$$

We exclude the case $\varphi(r_0) = (c_1 r_0 + c_2)^2$, since this corresponds to the constant curvature case of Section 4.3.2. Generally, this H_0 has only 3 symmetries ($\mathfrak{g}_1$), but in the case $\varphi = c_1(r_0^2 - 1)$ it has a fourth symmetry and corresponds to the case $H_0 = H_{14}$ of (4.4), with symmetry algebra $\mathfrak{g}_1 + \mathfrak{g}_4$.

For generic $\varphi(r_0)$, the kinetic energy H_0 has the symmetry algebra $\mathfrak{g}_1$. It is easy to check that H_0, $\mathcal{C}_1$ (of (2.4)) and K^2 (for any element K of $\mathfrak{g}_1$) are functionally independent, so we use these to construct some associated involutive systems, with

$$H = H_0 + h, \qquad F_1 = \mathcal{C}_1 + g_1, \qquad F_2 = K^2 + g_2, \tag{5.5}$$

where h, g_1, g_2 are arbitrary functions of q_1, q_2, q_3.

We just present the results. The calculations are straightforward.

The Case $K = e_1$

Involutivity of (5.5) leads to

$$h = \varphi_1(r_0) + \frac{q_1^2 \varphi(r_0)}{q_1^2 - q_3^2} \big(\varphi_2\big(q_3^2 - q_1^2\big) + \big(q_3^2 - q_1^2\big)\varphi_3(q_2)\big),$$
$$g_1 = \varphi_2\big(q_3^2 - q_1^2\big) + \big(q_3^2 - q_1^2\big)\varphi_3(q_2), \qquad g_2 = \varphi_3(q_2), \tag{5.6}$$

which gives the separation variables $u = \frac{q_3}{q_1}$, $v = q_3^2 - q_1^2$, $w = q_2$, in terms of which

$$H = \big(u^2 - 1\big)\varphi(u)p_u^2 + \varphi_1(u) - \frac{\varphi(u)}{u^2 - 1}F_1,$$
$$F_1 = 4v^2 p_v^2 + \varphi_2(v) + vF_2, \qquad F_2 = p_w^2 + \varphi_3(w).$$

Remark 5.2 (the involution ι_1). We can use the involution ι_1 to transform this system to an equivalent one for which $K = f_1$.

The case $K = h_1$

Involutivity of (5.5) leads to

$$h = \varphi_1(r_0) + \frac{q_1^2 \varphi(r_0)}{q_1^2 - q_3^2}\left(\varphi_2\left(\frac{q_3^2 - q_1^2}{q_2^2}\right) + \frac{q_1^2 - q_3^2}{4\big(q_1^2 - q_2^2 - q_3^2\big)}\varphi_3\big(q_1^2 - q_2^2 - q_3^2\big)\right),$$
$$g_1 = \varphi_2\left(\frac{q_3^2 - q_1^2}{q_2^2}\right) + \frac{q_1^2 - q_3^2}{4\big(q_1^2 - q_2^2 - q_3^2\big)}\varphi_3\big(q_1^2 - q_2^2 - q_3^2\big), \qquad g_2 = \varphi_3\big(q_1^2 - q_2^2 - q_3^2\big), \tag{5.7}$$

which gives the separation variables $u = \frac{q_3}{q_1}$, $v = \frac{q_3^2 - q_1^2}{q_2^2}$, $w = q_1^2 - q_2^2 - q_3^2$, in terms of which

$$H = (u^2 - 1)\varphi(u)p_u^2 + \varphi_1(u) - \frac{\varphi(u)}{u^2 - 1}F_1,$$
$$F_1 = 4v^2(v + 1)p_v^2 + \varphi_2(v) + \frac{v}{4(v + 1)}F_2, \qquad F_2 = 16w^2 p_w^2 + \varphi_3(w).$$

Remark 5.3 (the involution ι_1). This system is invariant under the action of the involution ι_1, up to a relabelling of φ_3, since $q_1^2 - q_2^2 - q_3^2 \mapsto \frac{1}{q_1^2 - q_2^2 - q_3^2}$.

5.3.1 The conformally flat case H_{14} of (4.4)

The kinetic energy $H_0 = H_{14}$ of (4.4) is a specific example of H_0 of (5.4), corresponding to $\varphi(r_0) = 1 - r_0^2$, and has the 4-dimensional symmetry algebra $\mathfrak{g}_1 + \mathfrak{g}_4$, with basis e_1, h_1, f_1, h_4, with h_4 commuting with the whole of $\mathfrak{g}_1$. Consequently the cases (5.6) and (5.7) simply reduce to this choice of $\varphi(r_0)$. However, there are additional possibilities involving the element h_4. Since $H_0 = \mathcal{C}_1 - \frac{1}{16}h_4^2$, we cannot use (5.5), with $K = h_4$. We can, however, use H_0, h_4 and any element K of $\mathfrak{g}_1$.

The commuting pair h_4, e_1

With the choice

$$F_1 = \frac{1}{16}h_4^2 + g_1(\boldsymbol{q}), \qquad F_2 = e_1^2 + g_2(\boldsymbol{q}),$$

a simple calculation leads to

$$h = \varphi_3\big(q_3^2 - q_1^2\big) + \big(q_3^2 - q_1^2\big)\varphi_2(q_2) - \varphi_1\left(\frac{q_3}{q_1}\right),$$

depending upon 3 arbitrary functions, with $g_1 = \varphi_1\left(\frac{q_3}{q_1}\right)$ and $g_2 = \varphi_2(q_2)$.

We can use ι_1 to derive an equivalent system with e_1 replaced by f_1.

The commuting pair h_4, h_1

With the choice

$$F_1 = \frac{1}{16}h_4^2 + g_1(\boldsymbol{q}), \qquad F_2 = \frac{1}{4}h_1^2 + g_2(\boldsymbol{q}),$$

a simple calculation leads to

$$h = \varphi_3\left(\frac{q_3^2 - q_1^2}{q_2^2}\right) + \frac{\big(q_3^2 - q_1^2\big)\varphi_2\big(q_2^2 + q_3^2 - q_1^2\big)}{q_2^2 + q_3^2 - q_1^2} - \varphi_1\left(\frac{q_3}{q_1}\right),$$

depending upon 3 arbitrary functions, with $g_1 = \varphi_1\big(\frac{q_3}{q_1}\big)$ and $g_2 = \varphi_2\big(q_2^2 + q_3^2 - q_1^2\big)$.

This system is invariant (up to a simple redefinition of φ_2) under the action of ι_1.

6   Adding potentials: super-integrability

In this section we consider the possibility of adding further integrals, F_3, F_4, to separable cases of Section 5, which can no longer be in involution with H, F_1, F_2, but should Poisson commute with H itself: $\{H, F_3\} = \{H, F_4\} = 0$. Having any additional integrals, the system is referred to as *super-integrable*. The functions should be chosen to be *functionally independent*, so the Jacobian matrix

$$\frac{\partial(H, F_i)}{\partial \boldsymbol{x}}, \qquad \text{where} \qquad \boldsymbol{x} = (q_1, \ldots, p_3),$$

has *maximal rank*. Whilst the maximal rank for a set of functions in this space is 6, the maximal rank for a set of *first integrals* is 5, since in this case, the level surface

$$\mathcal{S} = \{\boldsymbol{x} \colon H = c_0, \, F_i = c_i\}_{i=1}^4,$$

has dimension *one*, so represents an (unparameterised) trajectory of the dynamical system. A super-integrable system with the maximal number of functionally independent integrals is called *maximally super-integrable*. "Solving" the system of equations defining $\mathcal{S}$, gives the solution, but this cannot in general be determined explicitly. Being only 5 equation in a 6-dimensional space, this solution will depend upon a single additional parameter (as well as the parameters c_i), which will be some function of t, but not necessarily t itself.

If we start with a separable system of Section 5, depending upon 3 arbitrary, single-variable functions, then each additional integral imposes differential constraints on these arbitrary functions. Our maximally super-integrable systems depend on a finite number of arbitrary *parameters*, whose coefficients are *specific* functions (rational in our examples). The set of functions, $\{H, F_i\}_{i=1}^4$, will then generate a non-Abelian Poisson algebra, which may or may not be finite-dimensional.

To simplify all of these calculations, we choose F_3, F_4 to be a pair of functions whose leading order parts (in momenta) *commute*, but allow for the case $\{F_3, F_4\} \neq 0$, in which case

$$\{F_3, F_4\} = \sum_{i=1}^3 X_i(\boldsymbol{q})p_i$$

is a *first order* integral.

By choosing the leading order terms of each integral $\{F_i\}_{i=1}^4$ to be just K^2, for some element K of the symmetry algebra $\mathfrak{g}$, the automorphisms ι_1 and ι_{23} (where appropriate) of $\mathfrak{g}$ induce corresponding automorphisms of the Poisson algebra generated by $\{F_i\}_{i=1}^4$. This will be important when deriving the Poisson relations on the full Poisson algebra.

6.1   Constant curvature case of (4.4)

Here we have the symmetry algebra (4.3), with Casimir $H_0 = q_1^2(p_1^2 - p_2^2 - p_3^2)$. We start with the involutive system given in (5.2), with integrals

$$H = q_1^2(p_1^2 - p_2^2 - p_3^2) + h(\boldsymbol{q}), \qquad F_1 = e_1^2 + g_1(\boldsymbol{q}) = p_2^2 + g_1(\boldsymbol{q}),$$
$$F_2 = e_2^2 + g_2(\boldsymbol{q}) = p_3^2 + g_2(\boldsymbol{q}),$$

where the potential functions, $h(\boldsymbol{q})$, $g_1(\boldsymbol{q})$, $g_2(\boldsymbol{q})$, are given by (5.2b). We then add two further functions

$$F_3 = f_1^2 + g_3(\boldsymbol{q}) \qquad \text{and} \qquad F_4 = \frac{1}{4}f_2^2 + g_4(\boldsymbol{q}),$$

where f_1, f_2 are defined in the list (4.3). When we impose the conditions $\{H, F_3\} = \{H, F_4\} = 0$, it is a simple calculation to derive the following solution:

$$h = q_1^2\left(\frac{k_1}{q_2^2} + \frac{k_2}{q_3^2}\right), \qquad g_1 = -\frac{k_1}{q_2^2}, \qquad g_2 = -\frac{k_2}{q_3^2},$$
$$g_3 = -\frac{k_1(q_2^2 + q_3^2 - q_1^2)^2}{q_2^2}, \qquad g_4 = -\frac{k_2(q_2^2 + q_3^2 - q_1^2)^2}{q_3^2}. \tag{6.1}$$

In this case, we also find that h_1 is a first integral.

We therefore have 6 first integrals $(H, F_1, F_2, F_3, F_4, h_1)$, but the rank of the Jacobian is 5, so there should be an algebraic relation between them. Nevertheless, we consider these 6 functions as generators of our Poisson algebra. Under the action of the involutions (automorphisms of the symmetry algebra) ι_1 and ι_{23}, we have

$$\iota_1 \colon \ (H, F_1, F_2, F_3, F_4, h_1, k_1, k_2) \mapsto (H, F_3, F_4, F_1, F_2, -h_1, k_1, k_2),$$
$$\iota_{23} \colon \ (H, F_1, F_2, F_3, F_4, h_1, k_1, k_2) \mapsto (H, F_2, F_1, F_4, F_3, h_1, k_2, k_1),$$

so the entire Poisson algebra should obey such symmetry rules. We can use this in the derivation of the Poisson algebra. For example, if we know the formula for $\{F_1, F_3\}$, then we can use ι_{23} to deduce the formula for $\{F_2, F_4\}$. Whenever we introduce a new element of our algebra, we should simultaneously introduce any new elements which are derived through the action of these involutions. In this way we add a further 5 elements, which, by construction, satisfy $\{H, F_i\} = 0$, for all i.

Some Poisson relations are very simple to derive

$$\{F_1, F_2\} = \{F_3, F_4\} = 0 \qquad \text{and} \qquad \{F_i, h_1\} = \lambda_i F_i, \qquad i = 1, \dots, 9,$$

where $\lambda = (4, 4, -4, -4, 0, 0, 4, 0, -4)$.

The Poisson brackets $\{F_1, F_3\}$, $\{F_1, F_4\}$, $\{F_2, F_3\}$, $\{F_2, F_4\}$, are all cubic in momenta and could be *linear* combinations of $\{h_1 F_i, h_1 H, h_1^3\}_{i=1}^4$, but this is not the case. However, note that $\{F_1, F_3\}$ and $\{F_2, F_4\}$ are related through the involutions, as are $\{F_1, F_4\}$ and $\{F_2, F_3\}$. We can define two new *quadratic* elements F_5, F_6 through the relations

$$\{F_1, F_3\} = h_1(h_1^2 - 4H - 4F_5 - 4k_1), \qquad \{F_2, F_4\} = h_1(h_1^2 - 4H - 4F_6 - 4k_2),$$

with $F_5 \leftrightarrow F_6$ under ι_{23}. These functions can be written

$$F_5 = q_3^2 F_1 + q_1^2 F_2 + q_3 p_3(q_3 p_3 + 2q_1 p_1), \qquad F_6 = q_2^2 F_2 + q_1^2 F_1 + q_2 p_2(q_2 p_2 + 2q_1 p_1).$$

We define F_7, F_9 by the equations

$$\{F_1, F_6\} = 2h_1 F_1 + 4F_7, \qquad \{F_3, F_6\} = -2h_1 F_3 + 4F_9,$$

related by $F_7 \leftrightarrow F_9$ under ι_1.

The function F_8 is defined by the second of the following equations

$$\{F_1, F_4\} + \{F_2, F_3\} = 8h_1\left(H + F_5 + F_6 - \frac{1}{4}h_1^2\right), \qquad \{F_1, F_4\} - \{F_2, F_3\} = 16F_8,$$

after which, we find that $\{F_5, F_6\} = 4F_8$.

The action of the two involutions is then given by:

|	H	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	h_1	k_1	k_2
ι_1:	H	F_3	F_4	F_1	F_2	F_5	F_6	F_9	F_8	F_7	$-h_1$	k_1	k_2
ι_{23}:	H	F_2	F_1	F_4	F_3	F_6	F_5	$-F_7$	$-F_8$	$-F_9$	h_1	k_2	k_1

The action of ι_{23} on $\{F_1, F_6\}$ and $\{F_3, F_6\}$ then gives

$$\{F_2, F_5\} = 2h_1 F_2 - 4F_7, \qquad \{F_4, F_5\} = -2h_1 F_4 - 4F_9.$$

This phenomenon of connecting four different Poisson relations through the involutions is depicted in Fig. 1(a), where we define $P_{ij} = \{F_i, F_j\}$ (see Table 4). Sometimes only *two* relations are connected, such as with P_{13} and P_{24} (Fig. 1(b)), or even just *one*, such as with P_{79}, because of invariance properties.

Remark 6.1 (commutativity)**.** The actions of ι_1 and ι_{23} commute on these functions.

After inputting these known brackets, it is possible to use the Jacobi identity to derive all the others. The full set of Poisson relations is given in Table 4, with the array $P = (P_{ij})$, using the order $(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10} = h_1)$. The lower part of the matrix is given by
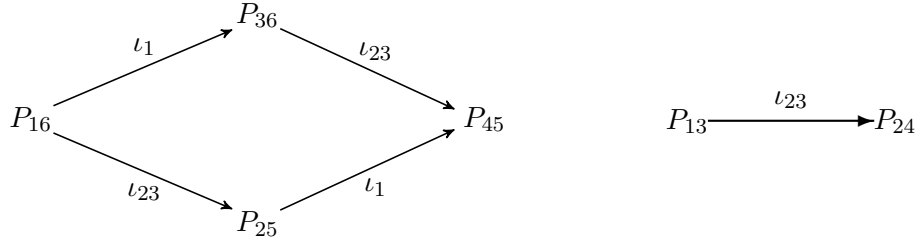
(a) Four connected bracket relations.   (b) Two connected bracket relations.

Figure 1. Bracket relations connected through ι_1 and ι_{23}.

skew-symmetry. In this context, H is just a parameter, since it commutes with all 10 elements. We find

$$P_{13} = h_1\big(h_1^2 - 4(H + F_5 + k_1)\big),$$

$$P_{14} = h_1\big(4(H + F_5 + F_6) - h_1^2\big) + 8F_8,$$

$$P_{57} = P_{18} = \frac{1}{2}\big((F_1 + F_2)\big(4H + 4F_5 - h_1^2\big) + 4(F_1 F_6 + k_1 F_2)\big),$$

$$P_{19} = 2(2(F_5 + H)(2(F_6 + H) + F_5) - F_1 F_4) - h_1^2(3F_5 + 2F_6 + 4H - k_2) + \frac{1}{2}h_1^4$$
$$\qquad - 4(k_2(F_5 + H + k_1) + F_5 k_1),$$

$$P_{58} = H(4F_6 - h_1^2 + 4H) + \frac{1}{2}F_5\big(8F_6 - h_1^2 + 12H\big) + 2F_5^2 - 2(k_1 + k_2)F_5$$
$$\qquad + \frac{1}{2}k_2\big(h_1^2 - 4H\big) - 2k_1 k_2,$$

$$P_{78} = \frac{1}{4}h_1(F_1 + F_2)\big(h_1^2 - 4(H + F_5 + F_6)\big).$$

From the list given above, we can derive all except P_{79} by using the involutions (as in Fig. 1). The relevant groupings are

$$(P_{13}, P_{24}), \qquad (P_{14}, P_{23}), \qquad (P_{18}, P_{28}, P_{38}, P_{48}), \qquad (P_{19}, P_{29}, P_{37}, P_{47}),$$
$$(P_{57}, P_{59}, P_{67}, P_{69}), \qquad (P_{58}, P_{68}), \qquad (P_{78}, P_{89}).$$

For example, applying ι_{23} to the formula for P_{13}, we get $P_{24} = h_1(h_1^2 - 4(H + F_6 + k_2))$. The equality $P_{57} = P_{18}$ follows from the Jacobi identity for the elements F_1, F_5, F_6. The most complicated entry in matrix P is P_{79}, which is not obtainable in this way, since it is invariant (up to a sign) under both involutions:

$$P_{79} = 2(F_2 F_9 - F_4 F_7) + \frac{1}{4}h_1\big(4(H + F_5 + F_6) - h_1^2\big)\big(h_1^2 - 4H - 4F_5 - 2F_6\big)$$
$$\qquad + \frac{1}{2}k_1 h_1\big(4(H + F_5 + F_6) - h_1^2\big) + k_2 h_1\big(4(H + F_5) - h_1^2\big) + 4h_1 k_1 k_2.$$

Under ι_1, $P_{79} \mapsto -P_{79}$, as it should. Under ι_{23}, we should have $P_{79} \mapsto P_{79}$, but, in fact, $P_{79} \mapsto P_{79} + I_{79}$, where

$$I_{79} = 2(F_3 + F_4)F_7 - 2(F_1 + F_2)F_9 + \frac{1}{2}h_1(F_5 - F_6)\big(4(H + F_5 + F_6 - k_1 - k_2) - h_1^2\big)$$
$$\qquad + \frac{1}{2}(k_1 - k_2)h_1\big(4H - h_1^2\big),$$

which satisfies $I_{79} \mapsto -I_{79}$ under both involutions. However, this does not pose a contradiction, since in the explicit form of the Poisson algebra, $I_{79} = 0$.

Table 4. The 10-dimensional Poisson algebra $\{F_i, F_j\} = P_{ij}$.

$$P = \begin{pmatrix} 0 & 0 & P_{13} & P_{14} & 0 & 4F_7 + 2F_1 h_1 & 2F_1 F_2 & P_{18} & P_{19} & 4F_1 \\ & 0 & P_{23} & P_{24} & 2F_2 h_1 - 4F_7 & 0 & -2F_1 F_2 & P_{28} & P_{29} & 4F_2 \\ & & 0 & 0 & 0 & 4F_9 - 2F_3 h_1 & P_{37} & P_{38} & 2F_3 F_4 & -4F_3 \\ & & & 0 & -4F_9 - 2F_4 h_1 & 0 & P_{47} & P_{48} & -2F_3 F_4 & -4F_4 \\ & & & & 0 & 4F_8 & P_{57} & P_{58} & P_{59} & 0 \\ & & & & & 0 & P_{67} & P_{68} & P_{69} & 0 \\ & & & & & & 0 & P_{78} & P_{79} & 4F_7 \\ & & & & & & & 0 & P_{89} & 0 \\ & & & & & & & & 0 & -4F_9 \\ & & & & & & & & & 0 \end{pmatrix},$$

The functions (H, F_1, F_2, F_3, F_4) are functionally independent integrals of the Hamiltonian system, with Hamiltonian H. The functions $(F_5, F_6, F_7, F_8, F_9, h_1)$ must therefore satisfy six functionally independent relations. These can be obtained by looking at the full set of Jacobi identity relations, some of which have non-trivial entries, all of which vanish when the functional forms of F_i are inserted. We label such functions by the Jacobi identity which gave rise to them, so J_{ijk} is (up to an overall constant multiple) the entry corresponding to F_i, F_j, F_k. These also belong to "families", which are related through the action of the involutions. Six such functions are

$$J_{159} = 2F_3 F_7 - 4F_5 F_8 + 2F_1 F_9 - 4F_8 H + F_8 h_1^2 + 4k_1 F_8,$$
$$J_{269} = 2F_4 F_7 - 4F_6 F_8 + 2F_2 F_9 - 4F_8 H + F_8 h_1^2 + 4k_2 F_8,$$
$$J_{137} = 2F_7 \left(h_1^2 + 4H - 4k_1 + 4F_5\right) - 8F_1 F_8$$
$$\qquad + 4h_1 (F_2 H + F_2 k_1 + F_1 k_2 + (F_1 + F_2) F_5) - F_2 h_1^3,$$
$$J_{139} = 2F_9 \left(h_1^2 + 4H - 4k_1 + 4F_5\right) - 8F_3 F_8$$
$$\qquad - 4h_1 (F_4 H + F_4 k_1 + F_3 k_2 + (F_3 + F_4) F_5) + F_4 h_1^3,$$
$$J_{247} = -2F_7 \left(h_1^2 + 4H - 4k_2 + 4F_6\right) + 8F_2 F_8$$
$$\qquad + 4h_1 (F_1 H + F_1 k_2 + F_2 k_1 + (F_1 + F_2) F_6) - F_1 h_1^3,$$
$$J_{249} = -2F_9 \left(h_1^2 + 4H - 4k_2 + 4F_6\right) + 8F_4 F_8$$
$$\qquad - 4h_1 (F_3 H + F_3 k_2 + F_4 k_1 + (F_3 + F_4) F_6) + F_3 h_1^3.$$

These functions satisfy $J_{ijk} = 0$, and their Jacobian (with respect to the functions H, F_i) has rank 6, thus giving us the necessary six relations on our algebra.

Remark 6.2 (action of the involutions). J_{159} and J_{269} are invariant under ι_1 and transform into one-another (up to sign) under ι_{23}. J_{137}, J_{139}, J_{247} and J_{249} are also connected, as depicted in Fig. 2.
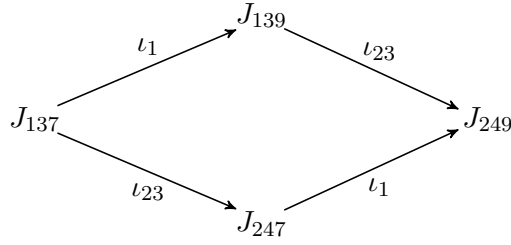
The very simple relation

$$J_{136} = F_2 F_3 - F_1 F_4 + 2F_8 h_1,$$

also exists and is invariant (up to sign) under the action of both involutions.

Remark 6.3 (comparison with the literature). The Hamiltonian H, with potential h given in (6.1), can be written

$$H = q_1^2 \left(p_1^2 - p_2^2 - p_3^2 + \frac{k_1}{q_2^2} + \frac{k_2}{q_3^2} \right),$$

Figure 2. Constraints connected through ι_1 and ι_{23}.

which is in the form of a Stäckel transform to a flat metric, so can be compared with the classification given in [6]. We can compare with the list of non-degenerate potentials (albeit in the quantum case with H_0 replaced by a Laplacian) given in [6, Section 5]. Allowing for the fact that this classification is for the Euclidean case and involves some complex coordinates, the above potential can be considered as a reduction of the 5 parameter potential $V_{[2,1,1,1]}$ of [6]; it is, in fact a reduced case, with our parameters (k_1, k_2) corresponding to their (a_2, a_3).

6.2 The flat case of (4.6)

The simplest flat case, with H given by (5.3), has integrals of the form

$$F_1 = e_1^2 + g_1, \qquad F_2 = 4e_2^2 + g_2, \qquad F_3 = f_1^2 + g_3, \qquad F_4 = f_2^2 + g_4,$$

since this is invariant under the action of the involution ι_1, so the Poisson algebra possesses this automorphism. This is the only case we consider here.

The functions φ_1, φ_2 of Section 5.2.2 are constrained by the additional integrals, giving

$$h = \frac{k_1(q_1 - q_3)^2}{q_2^2} + \frac{2k_2 q_1}{q_1 - q_3}, \qquad g_1 = -\frac{k_1}{q_2^2}, \qquad g_2 = -\frac{k_2}{(q_1 - q_3)^2},$$

$$g_3 = -\frac{k_1\left(q_1^2 - q_2^2 - q_3^2\right)^2}{q_2^2}, \qquad g_4 = -\frac{k_2\left(q_1^2 - q_2^2 - q_3^2\right)^2}{(q_1 - q_3)^2}. \tag{6.2}$$

In this case, we also find that h_1 is a first integral.

We therefore have 6 first integrals $(H, F_1, F_2, F_3, F_4, h_1)$, but the rank of the Jacobian is 5, so there should be an algebraic relation between them. Nevertheless, we consider these 6 functions as generators of our Poisson algebra. Under the action of the involution (automorphism of the symmetry algebra) ι_1, we have

$$\iota_1 \colon \ (H, F_1, F_2, F_3, F_4, h_1, k_1, k_2) \mapsto (H, F_3, F_4, F_1, F_2, -h_1, k_1, k_2),$$

so the entire Poisson algebra should obey such symmetry rules, induced by the definitions below.

We have $\{F_1, F_2\} = \{F_3, F_4\} = 0$ and the cubic expression $\{F_1, F_3\} = 4h_1 F_5$ *factorises*, giving us a new *quadratic* integral F_5, which is invariant under the action of ι_1. Two new *cubic* integrals (related through ι_1) are defined by

$$\{F_1, F_4\} = 8F_6 \qquad \text{and} \qquad \{F_2, F_3\} = 8F_7,$$

whilst $\{F_2, F_4\}$ is just *linear*: $\{F_2, F_4\} = -8k_2 h_1$. The first four brackets with F_5 are

$$\{F_1, F_5\} = 2h_1 F_1, \qquad \{F_2, F_5\} = 4F_8, \qquad \{F_3, F_5\} = -2h_1 F_3, \qquad \{F_4, F_5\} = 4F_9,$$

giving us two new *cubic* integrals. We find the following factorisation:

$$F_6 + F_7 = h_1 F_{10},$$

which defines another *quadratic* integral. All remaining brackets can be determined in terms of the above F_i and h_1. In these relations, H acts as a parameter, since it (by definition) commutes with *all* F_i. The action of the involution ι_1 is then given by:

	H	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	h_1
$\iota_1:$	H	F_3	F_4	F_1	F_2	F_5	$-F_7$	$-F_6$	F_9	F_8	F_{10}	$-h_1$

The integral h_1 acts diagonally:

$$\{F_i, h_1\} = \lambda_i F_i, \qquad i = 1, \ldots, 10, \qquad \text{where} \qquad \lambda = (4, 4, -4, -4, 0, 0, 0, 4, -4, 0).$$

Defining $P_{ij} = \{F_i, F_j\}$, the remaining independent entries in the Poisson matrix are

$$P_{16} = 2F_1(H + 3F_{10} - k_2) + 4k_1F_2, \qquad P_{17} = 2(F_2F_5 + F_8h_1), \qquad P_{18} = 2F_1F_2,$$
$$P_{19} = -4F_5(H + 2F_{10} - k_2) - 2F_1F_4 - 8k_1F_{10}, \qquad P_{110} = 4F_8, \qquad P_{26} = -4k_2F_1,$$
$$P_{27} = 4(F_2F_{10} + k_2F_1), \qquad P_{28} = 0, \qquad P_{29} = 4F_{10}(k_2 - H - F_{10}) + 8k_2(F_5 + 2k_1),$$
$$P_{210} = 0, \qquad P_{56} = 2(F_5(H + 2F_{10} - k_2) - F_6h_1 + 2k_1F_{10}),$$
$$P_{58} = F_2F_5 - F_8h_1 - 2k_1F_2 - F_1(H + 3F_{10} - k_2), \qquad P_{510} = 2(F_7 - F_6),$$
$$P_{67} = F_4F_8 - F_2F_9 + h_1(F_{10}(H - k_2) - 2k_2(F_5 + 2k_1) + F_{10}^2),$$
$$P_{68} = 2(F_2F_6 + F_8(-2F_{10} - H + k_2)), \qquad P_{69} = 2F_9(2F_{10} + H - k_2) - 2F_4F_6,$$
$$P_{610} = 2(F_{10}(H - k_2) - 2k_2(F_5 + 2k_1) + F_{10}^2),$$
$$P_{89} = -2h_1(F_{10}H - F_{10}k_2 + 2F_{10}^2 - 4k_1k_2), \qquad P_{810} = 4F_1k_2 + 2F_2F_{10}.$$

The remaining entries can be obtained by using the involution ι_1.

The functions (H, F_1, F_2, F_3, F_4) are functionally independent integrals of the Hamiltonian system, with Hamiltonian H. The functions $(F_5, F_6, F_7, F_8, F_9, F_{10}, h_1)$ must therefore satisfy seven functionally independent relations. Again, these can be obtained by looking at the full set of Jacobi identity relations, some of which have non-trivial entries, all of which vanish when the functional forms of F_i are inserted. We label such functions by the Jacobi identity which gave rise to them, so J_{ijk} is (up to an overall constant multiple) the entry corresponding to F_i, F_j, F_k. These also arise in "pairs", which are related through the action of the involution ι_1. The first, in fact, comes from the *definition* of F_{10} (which satisfies $I_{10} \mapsto -I_{10}$ under the action of ι_1):

$$I_{10} = F_6 + F_7 - h_1F_{10}, \qquad J_{1310} = F_2F_3 - F_1F_4 + (F_6 - F_7)h_1,$$
$$J_{267} = F_1(F_{10} + H - k_2) + F_2(F_5 + 2k_1) - F_8h_1,$$
$$J_{467} = F_3(F_{10} + H - k_2) + F_4(F_5 + 2k_1) + F_9h_1,$$
$$J_{146} = F_4F_8 - F_6(F_{10} + H - k_2) + k_2h_1(F_5 + 2k_1),$$
$$J_{237} = F_2F_9 + F_7(F_{10} + H - k_2) - k_2h_1(F_5 + 2k_1),$$
$$J_{1610} = F_2F_6 - F_8(F_{10} + H - k_2) + k_2h_1F_1.$$

6.2.1 Flat coordinates

We saw that the flat coordinates (3.18) reduce this H_0 to the form (3.19). In these coordinates, the first 5 (functionally independent) integrals of our Poisson algebra take the form

$$H = 2P_1P_3 - P_2^2 + \frac{k_1}{Q_2^2} + k_2(1 + Q_2^2 - 2Q_1Q_3),$$
$$F_1 = 4\left(Q_2^2P_1^2 + Q_3^2\left(P_2^2 - \frac{k_1}{Q_2^2}\right) + 2Q_2Q_3P_1P_2\right),$$

$$F_2 = 4\big(P_1^2 - k_2 Q_3^2\big), \qquad F_3 = (Q_1 P_2 + Q_2 P_3)^2 - k_1 \frac{Q_1^2}{Q_2^2}, \qquad F_4 = P_3^2 - k_2 Q_1^2.$$

The functions $F_5, \ldots, F_{10}$ can similarly be found and since the transformation is canonical, the Poisson relations do not change.

In these coordinates, the involution ι_1 is generated by $S = -\big(2q_3 P_1 + q_2 P_2 + \frac{1}{2} q_1 P_3\big)$.

Remark 6.4 (comparison with the literature). The Hamiltonian H, with potential h given in (6.2), can be written

$$H = (q_1 - q_3)^2 \left(p_1^2 - p_2^2 - p_3^2 + \frac{k_1}{q_2^2} + \frac{2k_2 q_1}{(q_1 - q_3)^3} \right),$$

which is in the form of a Stäckel transform to a flat metric, so, again, can be compared with the classification given in [6]. We can compare with the list of non-degenerate potentials (albeit in the quantum case with H_0 replaced by a Laplacian) given in of [6, Section 5]. Allowing for the fact that this classification is for the Euclidean case and involves some complex coordinates, the above potential can be considered as a reduction of the 5 parameter potential $V_{[2,2,1]}$ of [6]; it is, in fact a reduced case, with our parameters (k_1, k_2) corresponding to their (a_1, a_3).

In the flat coordinates (diagonalised), it again corresponds to $V_{[2,2,1]}$, but now with (k_1, k_2) corresponding to their (a_1, a_4).

6.3 The conformally flat case (5.4)

In this section we consider the involutive system (5.4), with $K = e_1$, giving the potential functions (5.6). To simplify the calculations of this section, we choose the specific metric coefficient $\varphi(r_0) = r_0$, leading to

$$H = H_0 + h(\boldsymbol{q}), \qquad \text{where} \qquad H_0 = q_1 q_3 \big(p_1^2 - p_2^2 - p_3^2\big),$$

whose Ricci tensor is non-constant: $R = \frac{5\big(q_1^2 - q_3^2\big)}{2q_1 q_3}$. In addition to F_1, F_2, given in (5.4) and (5.6), we require that the quadratic function

$$F_3 = \frac{1}{4} h_1^2 + g_3(\boldsymbol{q}),$$

satisfies $\{F_3, H\} = 0$, which restricts the component functions of $h(\boldsymbol{q})$, as well as determining g_3, but leaves $\varphi_1\big(\frac{q_3}{q_1}\big)$ *arbitrary*. Specifically, we find

$$\varphi_2\big(q_3^2 - q_1^2\big) = k_0 + k_1\big(q_3^2 - q_1^2\big) + k_2\big(q_3^2 - q_1^2\big)^2, \qquad \varphi_3(q_2) = k_2 q_2^2 - \frac{k_3}{2q_2^2},$$

$$g_3 = -\big(q_1^2 - q_2^2 - q_3^2\big)\big(k_1 - k_2\big(q_1^2 - q_2^2 - q_3^2\big)\big).$$

Setting $\varphi_1\big(\frac{q_3}{q_1}\big) = 0$ (since it effectively an additive constant), we therefore have

$$H = q_1 q_3 \left(p_1^2 - p_2^2 - p_3^2 + \frac{k_0}{q_1^2 - q_3^2} - k_1 + k_2\big(q_1^2 - q_2^2 - q_3^2\big) + \frac{k_3}{2q_2^2} \right),$$

$$F_1 = \mathcal{C}_1 - k_1\big(q_1^2 - q_3^2\big) + k_2\big(q_1^2 - q_3^2\big)\big(q_1^2 - q_2^2 - q_3^2\big) + \frac{k_3\big(q_1^2 - q_3^2\big)}{2q_2^2},$$

$$F_2 = e_1^2 + k_2 q_2^2 - \frac{k_3}{2q_2^2}, \qquad F_3 = \frac{1}{4} h_1^2 - \big(q_1^2 - q_2^2 - q_3^2\big)\big(k_1 - k_2(q_1^2 - q_2^2 - q_3^2)\big).$$

The algebra of these three integrals is easily calculated:

$$\{F_1, F_2\} = 0, \qquad \{F_1, F_3\} = 0, \qquad \{F_2, F_3\} = 2F_4,$$

where F_4 is a cubic expression

$$F_4 = F_2 h_1 - 2k_1 q_2 p_2 + 4k_2 q_2 \big(q_2(q_1 p_1 + q_3 p_3) + (q_1^2 - q_3^2)p_2\big),$$

which cannot be written as a polynomial in H and its integrals, but does satisfy the algebraic relation

$$F_4^2 = 4F_2^2 F_3 + 4k_1 F_2(F_3 - F_1) - 4k_2(F_3 - F_1)^2 + 4k_2 k_3(F_3 + F_1)$$
$$+ 2k_1 k_3 F_2 + k_3(2k_1^2 - k_2 k_3).$$

We can use this to derive the formulae for $\{F_i, F_4\}$: $\{F_1, F_4\} = 0$ and

$$\{F_2, F_4\} = 4F_2^2 + 4k_1 F_2 - 8k_2(F_3 - F_1) + 4k_2 k_3,$$
$$\{F_3, F_4\} = -8F_2 F_3 - 4k_1(F_3 - F_1) - 2k_1 k_3.$$

Remark 6.5. We have 4 functionally independent integrals, H, F_1, F_2, F_3, which form a closed algebra (with the inclusion of the functionally *dependent* F_4). In 3-dimensions, this system is super-integrable, but *not* maximally. Since we only have a 3-dimensional symmetry algebra $\mathfrak{g}_1$, we cannot build any further integrals out of this. However, since the metric is not constant curvature, it is possible that other integrals exist, which are not built in this way.

7 Conclusions

We started this paper by considering a specific 3-dimensional realisation of $\mathfrak{sl}(2)$ (our algebra $\mathfrak{g}_1$) and showed how to embed this into some 6- and 10-dimensional Lie algebras, with very specific structure. The Casimir functions (3.5) and (3.9) of the 6-dimensional algebras, represent the kinetic energy on manifolds with these symmetries, and the 10-dimensional algebra gave us the corresponding *conformal algebra*. This was used in Sections 5 and 6, where we considered some specific diagonal examples of these Casimir functions.

Indeed, the main aim of this paper was to build super-integrable systems (and the associated non-Abelian Poisson algebras) with a given kinetic energy, which itself has a high degree of symmetry. The approach is most suited to kinetic energies derived from constant curvature (including flat) manifolds, as mainly considered here. However, we saw that it is possible to apply the method when the isometry algebra is smaller, but, in this case, it may not be possible to build enough *independent* integrals for the system to be *maximally* super-integrable.

We just considered 3-dimensional manifolds in this paper. We used the structure of the symmetry algebra to construct involutive triples, giving us separable systems. We further used the structure of the symmetry algebra to select the leading parts of additional integrals to obtain super-integrable restrictions. We selected our integrals in such a way that the automorphisms of the symmetry algebra (realised as canonical transformations) could be extended to act simply on the non-Abelian Poisson algebra, which enabled us to find a *finite closure* of the Poisson algebra.

We did not consider the classification of super-integrable systems, such as can be found in the literature (see, for example, [6, 11, 14]), but our approach can be applied to any kinetic energy associated with a flat or constant curvature metric, which includes most physical systems.

In this paper we made several choices which simplified our calculations, leaving us with a number of open problems.

Not all realisations of $\mathfrak{sl}(2)$ are equivalent. Whilst 2-dimensional realisations were classified by Lie (see [2, Section 2]), no such classification exists for the 3-dimensional case. Within our general $\mathfrak{g}_1$, with f_1 given by (2.8b), there are at least *two* equivalence classes (corresponding to degenerate and non-degenerate Casimir functions), but we don't have a full classification of inequivalent cases. Given such a choice of $\mathfrak{g}_1$, the constructions of Sections 2.3, 3 and 4 could be carried out: the general problem is to find all 6- and 10-dimensional, nontrivial extensions of $\mathfrak{g}_1$, which are then to be used in the construction of separable and super-integrable systems.

We saw in Section 4 that the 10-dimensional algebra contains several subalgebras, whose Casimir functions correspond to conformally equivalent metrics. The (infinitesimal) conformal factors formed another representation space of the algebra $\mathfrak{g}_1$. We don't yet have the full classification of subalgebras of our 10-dimensional conformal algebra. For any subalgebra, the Casimir function, representing the corresponding kinetic energy, could be used in the context of our analysis of Sections 5 and 6.

In Section 5 we made the *simplest choice* of involutive triple. More general quadratic forms would clearly lead to more complicated calculations, but could lead to some interesting examples.

In Section 6 we chose additional integrals that would minimise the complexity of our calculations. Clearly there are many different choices which could lead to interesting systems and corresponding Poisson algebras. Even for the simple choices we made, the Poisson algebras were very complicated and we have little understanding of the general structure. The Poisson algebras satisfy many polynomial constraints, which can be used to simplify some of the Poisson relations, but what is the minimal set of generators of these constraints?

The current paper is about classical Poisson algebras, but a similar analysis can be carried out for the quantum case, where super-integrability should allow us to construct explicit eigenfunctions, as was found in 2-dimensions [7].

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References

[1] Cariglia M., Hidden symmetries of dynamics in classical and quantum physics, *Rev. Modern Phys.* **86** (2014), 1283–1333, arXiv:1411.1262.

[2] Clarkson P.A., Olver P.J., Symmetry and the Chazy equation, *J. Differential Equations* **124** (1996), 225–246.

[3] Daskaloyannis C., Ypsilantis K., Unified treatment and classification of superintegrable systems with integrals quadratic in momenta on a two-dimensional manifold, *J. Math. Phys.* **47** (2006), 042904, 38 pages, math-ph/0412055.

[4] Dubrovin B.A., Fomenko A.T., Novikov S.P., Modern geometry – methods and applications, Part I, The geometry of surfaces, transformation groups, and fields, *Graduate Texts in Mathematics*, Vol. 93, Springer-Verlag, New York, 1984.

[5] Eisenhart L.P., Riemannian geometry, *Princeton Landmarks in Mathematics*, Princeton University Press, Princeton, NJ, 1997.

[6] Escobar-Ruiz M.A., Miller Jr. W., Toward a classification of semidegenerate 3D superintegrable systems, *J. Phys. A: Math. Theor.* **50** (2017), 095203, 22 pages, arXiv:1611.02977.

[7] Fordy A.P., Quantum super-integrable systems as exactly solvable models, *SIGMA* **3** (2007), 025, 10 pages, math-ph/0702048.

[8] Fordy A.P., A note on some superintegrable Hamiltonian systems, *J. Geom. Phys.* **115** (2017), 98–103, arXiv:1601.03079.

[9] Fordy A.P., Scott M.J., Recursive procedures for Krall–Sheffer operators, *J. Math. Phys.* **54** (2013), 043516, 23 pages, arXiv:1211.3075.

[10] Gilmore R., Lie groups, Lie algebras, and some of their applications, Wiley, New York, 1974.

[11] Kalnins E.G., Kress J.M., Miller Jr. W., Fine structure for 3D second-order superintegrable systems: three-parameter potentials, *J. Phys. A: Math. Theor.* **40** (2007), 5875–5892.

[12] Landau L.D., Lifshitz E.M., Mechanics, Course of Theoretical Physics, Vol. 1, Mechanics, Pergamon Press, Oxford, 1976.

[13] Marshall I., Wojciechowski S., When is a Hamiltonian system separable?, *J. Math. Phys.* **29** (1988), 1338–1346.

[14] Miller Jr. W., Post S., Winternitz P., Classical and quantum superintegrability with applications, *J. Phys. A: Math. Theor.* **46** (2013), 423001, 97 pages, arXiv:1309.2694.

[15] Smirnov R.G., The classical Bertrand–Darboux problem, *J. Math. Sci.* **151** (2008), 3230–3244, math-ph/0604038.

[16] Whittaker E.T., A treatise on the analytical dynamics of particles and rigid bodies, Cambridge University Press, Cambridge, 1988.