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Eigenvalue Problems for Lamé's Differential Equation

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Abstract. The Floquet eigenvalue problem and a generalized form of the Wangerin eigenvalue problem for Lamé's differential equation are discussed. Results include comparison theorems for eigenvalues and analytic continuation, zeros and limiting cases of (generalized) Lamé–Wangerin eigenfunctions. Algebraic Lamé functions and Lamé polynomials appear as special cases of Lamé–Wangerin functions.

Key words: Lamé functions; singular Sturm–Liouville problems; tridiagonal matrices

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1 Introduction

The Lamé equation (Arscott [1, Chapter IX]) is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \big(h - \nu(\nu+1)k^2 \operatorname{sn}^2(z,k)\big)w = 0, \tag{1.1}$$

where $\operatorname{sn}(z,k)$ is the Jacobian elliptic function with modulus $k \in (0,1)$ (Whittaker and Watson [22, Chapter XXII]), $\nu \in \mathbb{R}$ and h is the eigenvalue parameter. This equation has regular singularities at the points $z = 2mK + \mathrm{i}(2n+1)K'$, where m, n are integers and $K = K(k)$ and $K' = K'(k)$ denote complete elliptic integrals. Various eigenvalue problems for the Lamé equation have been treated in the literature.

The Lamé equation is an even Hill's equation with fundamental period $2K$. The theory of Hill's equation is well-known; see, e.g., Arscott [1], Eastham [3] and Magnus and Winkler [14]. Results on the periodic eigenvalue problem specific for the Lamé equation with eigenfunctions satisfying $w(z+2K) = \pm w(z)$ can be found in [6, Section 15,5], [9, 10] and [18, Chapter 29]. These functions have many applications; see, e.g., [2]. In Section 2 of this paper we will consider the more general Floquet eigenvalue problem $w(z+2K) = \mathrm{e}^{\mathrm{i}\mu\pi}w(z)$. For the general Hill's equation this eigenvalue problem is treated in Eastham [3]. Some results on the Floquet eigenvalue problem specific for the Lamé equation can be found in Ince [8, Sections 7 and 8].

Wangerin [21] showed that Lamé's equation appears when Laplace's equation is separated in confocal cyclidic coordinates of revolution. Such coordinate systems can be found in Moon and Spencer [17] and in Miller [16]. They include flat-ring, flat-disk, bi-cyclide and cap-cyclide coordinates. An outline of Wangerin's results is given in [6, Section 15.1.3]. In order to obtain harmonic functions relevant for applications special solutions of the Lamé equation called Lamé–Wangerin functions were introduced; see Erdélyi [5], [6, p. 88]. The Lamé–Wangerin eigenvalue problem is obtained when we require that $(\operatorname{sn} z)^{1/2}w(z)$ stays bounded at the singularities $\mathrm{i}K'$ and $2K + \mathrm{i}K'$; see Erdélyi [5] and Erdélyi, Magnus and Oberhettinger [6, Section 15.6]. These eigenfunctions will be defined on the segment $(\mathrm{i}K', 2K + \mathrm{i}K')$ but can then be continued analytically.

In Section 3 of this paper we consider a more general eigenvalue problem whose eigenfunctions $w(z)$ have the form

$$w(z) = (z - \mathrm{i}K')^{\nu+1} \sum_{n=0}^{\infty} q_n (z - \mathrm{i}K')^{2n}$$

at $z = \mathrm{i}K'$ and a similar condition at $z = 2K + \mathrm{i}K'$. We call these eigenfunctions generalized Lamé–Wangerin functions. Every classical Lamé–Wangerin function is also a generalized Lamé–Wangerin function but not vice versa unless $\nu \geq -\frac{1}{2}$.

The motivation for introducing these functions is as follows. In Section 2 we show that the eigenvalues of the Floquet eigenvalue problem agree with the eigenvalues of an infinite tridiagonal matrix F (considered in the Hilbert sequence space $\ell^2(\mathbb{Z})$, $\mathbb{Z}$ the set of integers). One is especially interested in the case that a matrix entry in the diagonal above or below the main diagonal of F vanishes because then the eigenvalue problem splits in two problems whose eigenvalues are given by infinite tridiagonal submatrices of F that are only infinite in one direction (the underlying Hilbert space can be taken as $\ell^2(\mathbb{N}_0)$, $\mathbb{N}_0$ the set of non-negative integers). It turns out that this special case occurs if and only if $\nu+\mu$ or $\nu-\mu$ is an integer (ν is the parameter in Lamé's equation and μ is the parameter in Floquet's condition.) Interestingly, if $\nu + \mu$ or $\nu - \mu$ is an integer then the eigenvalues of one of the submatrices are identical with the eigenvalues of a classical Lamé–Wangerin problem. This is a simple observation but as far as I know has not been stated in the literature. Of course, the obvious question is: If the eigenvalues of one submatrix of F are those for a classical Lamé–Wangerin problem what is the meaning of the eigenvalues of the complementary submatrix? As we show in this paper, these are the eigenvalues for a generalized Lamé–Wangerin problem (non-classical except when $\nu = -\frac{1}{2}$).

A second motivation for introducing the generalized Lamé–Wangerin functions is as follows. Lamé polynomials and algebraic Lamé functions are not special cases of classical Lamé–Wangerin functions. However, they are special case of generalized Lamé–Wangerin function. We show in Sections 6 and 7 how Lamé polynomials and algebraic Lamé functions appear in the notation of generalized Lamé–Wangerin functions. We should mention that we adopt the name "algebraic Lamé functions" from [6, p. 68]. These functions are called "Lamé–Wangerin functions" in Lambe [13] and non-meromorphic Lamé functions in Finkel et al. [7].

In Section 5 we compare the eigenvalues of the Floquet and the Lamé–Wangerin problems. In Sections 6 and 7 we show that algebraic Lamé functions and Lamé polynomials are special cases of (generalized) Lamé–Wangerin functions. In Section 8 we investigate the number of zeros of Lamé–Wangerin eigenfunctions. In Section 9 we find the limit of Lamé–Wangerin functions as $k \to 0$.

I consider some of the results in this paper as new but not all results are new. The treatment of the generalized Lamé–Wangerin problem is new. The recursions (3.9) and (3.17) are known from [6] but the "symmetric" recursions (3.12), (3.19) appear to be new. The latter recursions are used in some proofs and also in Section 6. The results in Sections 4, 5, 8 and 9 are new. Lamé polynomials and algebraic Lamé functions are well-known, so I make no claim that Sections 6 and 7 contain new results.

2 Floquet solutions

On the real axis $z \in \mathbb{R}$, (1.1) is a Hill equation with fundamental period $2K$. Let $\mu \in \mathbb{R}$. We call h a Floquet eigenvalue if there exists a nontrivial solution w of (1.1) satisfying

$$w(z + 2K) = \mathrm{e}^{\mathrm{i}\pi\mu} w(z), \qquad z \in \mathbb{R}. \tag{2.1}$$

$w(z)$ is a corresponding Floquet eigenfunction. It is known [3, p. 31] that the eigenvalues are real and form a sequence converging to ∞. We denote the eigenvalues by

$$h_0(\mu,\nu,k) \le h_1(\mu,\nu,k) \le h_2(\mu,\nu,k) \le \cdots .$$

The eigenvalues are counted according to multiplicity. If μ is not an integer then

$$h_0(\mu,\nu,k) < h_1(\mu,\nu,k) < h_2(\mu,\nu,k) < \cdots .$$

Obviously, we have

$$h_m(\mu,\nu,k) = h_m(\mu+2,\nu,k) = h_m(-\mu,\nu,k) = h_m(\mu,-\nu-1,k). \tag{2.2}$$

Let $w_1(z,h,\nu,k)$ and $w_2(z,h,\nu,k)$ be the solutions of (1.1) satisfying the initial conditions $w_1(0)=1$, $\frac{\mathrm{d}w_1}{\mathrm{d}z}(0)=0$, $w_2(0)=0$, $\frac{\mathrm{d}w_2}{\mathrm{d}z}(0)=1$. Then Hill's discriminant D is given by

$$D(h,\nu,k) = w_1(2K,h,\nu,k) + \frac{\mathrm{d}w_2}{\mathrm{d}z}(2K,h,\nu,k).$$

The eigenvalues $h_m(\mu,\nu,k)$ are the solutions of the equation

$$D(h,\nu,k) = 2\cos(\mu\pi); \tag{2.3}$$

see [3, equation (2.4.4)]. From (2.3) we easily obtain the following result that will be needed later.

Theorem 2.1. *For every $m \in \mathbb{N}_0 = \{0,1,2,\dots\}$, the function $(\mu,\nu,k) \mapsto h_m(\mu,\nu,k)$ is continuous on $\mathbb{R}\times\mathbb{R}\times[0,1)$.*

Theorem 2.1 can also be inferred from results on Sturm–Liouville theory [12].

Following [6, p. 65] we transform (1.1) by setting

$$t = \tfrac{1}{2}\pi - \operatorname{am}(z,k), \tag{2.4}$$

where am is Jacobi's amplitude function. We note that (2.4) establishes a conformal mapping between the strip $|\Im z| < K'$ and the t-plane cut along the rays $m\pi \pm \mathrm{i}sL$, $s \ge 1$, $m \in \mathbb{Z}$, where

$$L := \operatorname{arccosh}\frac{1}{k} = \frac{1}{2}\ln\frac{1+k'}{1-k'}, \qquad k' = \sqrt{1-k^2}.$$

Then

$$\operatorname{sn} z = \cos t, \qquad \operatorname{cn} z = \sin t.$$

We obtain

$$\left(1-k^2\cos^2 t\right)\frac{\mathrm{d}^2w}{\mathrm{d}t^2} + k^2\cos t\sin t\frac{\mathrm{d}w}{\mathrm{d}t} + \left(h-\nu(\nu+1)k^2\cos^2 t\right)w = 0. \tag{2.5}$$

Since $\operatorname{am}(z+2K) = \operatorname{am} z + \pi$, condition (2.1) becomes

$$w(t+\pi) = \mathrm{e}^{-\mathrm{i}\pi\mu}w(t), \qquad t\in\mathbb{R}.$$

This condition is equivalent to $\mathrm{e}^{\mathrm{i}\mu t}w(t)$ being periodic with period π. Therefore, using Fourier series, eigenfunctions have the form

$$w(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{-\mathrm{i}(\mu+2n)t}. \tag{2.6}$$

By substituting (2.6) in (2.5), we obtain the three-term recursion

$$\rho_n c_{n-1} + (\sigma_n - h)c_n + \tau_{n+1}c_{n+1} = 0, \qquad n \in \mathbb{Z}, \tag{2.7}$$

where

$$\begin{aligned}
\rho_n &= -\tfrac{1}{4}k^2(2n-1+\mu+\nu)(2n-2+\mu-\nu),\\
\sigma_n &= \tfrac{1}{2}k^2\nu(\nu+1) + \left(1-\tfrac{1}{2}k^2\right)(2n+\mu)^2,\\
\tau_n &= -\tfrac{1}{4}k^2(2n+\mu+\nu)(2n-1+\mu-\nu).
\end{aligned}$$

This recursion is similar to the one given in [8, equation (7.1)] which is based on Fourier cosine series instead of the complex form of Fourier series we used. The behavior of solutions $\{c_n\}_{n\in\mathbb{Z}}$ of (2.7) as $n \to \infty$ is given by Perron's rule [19]. If $k \in (0,1)$ we choose n_0 so large that $\rho_n \neq 0$ and $\tau_{n+1} \neq 0$ for $n \geq n_0$. Then the solutions $\{c_n\}_{n>n_0}$ of equations (2.7) for $n \geq n_0$ form a two-dimensional vector space. There exists a recessive solution which is uniquely determined up to a constant factor with the property

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \frac{1-k'}{1+k'} < 1. \tag{2.8}$$

Every solution which is linearly independent of this solution satisfies

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \frac{1+k'}{1-k'} > 1.$$

Similar results hold for $n \to -\infty$. We obtain the following theorem.

Theorem 2.2. *Let $\mu, \nu \in \mathbb{R}$ and $k \in (0,1)$. Then h is one of the eigenvalues $h_m(\mu,\nu,k)$ if and only if the recursion (2.7) has a nontrivial solution $\{c_n\}_{n\in\mathbb{Z}}$ such that*

a) either there is n_0 such that $c_n = 0$ for $n \geq n_0$ or $\{c_n\}$ is recessive as $n \to \infty$; and

b) either there is n_0 such that $c_n = 0$ for $n \leq n_0$ or $\{c_n\}$ is recessive as $n \to -\infty$.

The expansion (2.6) of a corresponding eigenfunction converges in the strip $|\Im t| < L$.

Of course, a nontrivial solution $\{c_n\}$ of (2.7) can be zero for $n \geq n_0$ or $n \leq n_0$ only when one of the numbers ρ_n or τ_n vanishes. This happens if and only if at least one of the numbers $\mu \pm \nu$ is an integer. These interesting cases will be discussed in Sections 5, 6 and 7.

Alternatively, we may expand

$$w(t) = \left(1 - k^2\cos^2 t\right)^{1/2} \sum_{n=-\infty}^{\infty} d_n e^{-i(\mu+2n)t}.$$

Then we obtain the "adjoint" recursion

$$\tau_n d_{n-1} + (\sigma_n - h)d_n + \rho_{n+1}d_{n+1} = 0, \qquad n \in \mathbb{Z}. \tag{2.9}$$

Theorem 2.2 also holds with (2.9) in place of (2.7).

3 Generalized Lamé–Wangerin functions

A (classical) Lamé–Wangerin function $w(z)$ is a nontrivial solution of Lamé's equation (1.1) with the property that $(\operatorname{sn} z)^{1/2}w(z)$ stays bounded on the segment between the regular singularities K' and $2K+\mathrm{i}K'$; see [6, Section 15.6]. Such solutions exist only for specific values of h. If we substitute $z=u+\mathrm{i}K'$ then we obtain the singular Sturm–Liouville problem [23]

$$\frac{\mathrm{d}^2w}{\mathrm{d}u^2}+\big(h-\nu(\nu+1)\operatorname{sn}^{-2}(u,k)\big)w=0, \qquad 0<u<2K, \tag{3.1}$$

with the boundary condition that $(\operatorname{sn} u)^{-1/2}w$ stays bounded on $(0,2K)$.

The eigenvalue problem splits into two problems, one for functions that are even with respect to $K+\mathrm{i}K'$, that is,

$$w(K+\mathrm{i}K'+s)=w(K+\mathrm{i}K'-s) \qquad \text{for } -K<s<K, \tag{3.2}$$

and one for functions which are odd with respect to $K+\mathrm{i}K'$, that is,

$$w(K+\mathrm{i}K'+s)=-w(K+\mathrm{i}K'-s) \qquad \text{for } -K<s<K. \tag{3.3}$$

Without loss of generality, one may assume that $\nu\ge-\frac12$, and since the exponents at $\mathrm{i}K'$ and $2K+\mathrm{i}K'$ are $\{\nu+1,-\nu\}$, a Lamé–Wangerin function has the form

$$w(z)=(z-\mathrm{i}K')^{\nu+1}\sum_{n=0}^{\infty}q_n(z-\mathrm{i}K')^{2n} \tag{3.4}$$

for z close to $\mathrm{i}K'$ with $q_0\ne0$.

We generalize these eigenvalue problems as follows. Let $\nu\in\mathbb{R}$, $0<k<1$. We call $h\in\mathbb{C}$ an eigenvalue of the first Lamé–Wangerin problem if (1.1) admits a nontrivial solution w on the interval $(\mathrm{i}K',2K+\mathrm{i}K')$ which close to $z=\mathrm{i}K'$ has the form (3.4) and satisfies $w'(K+\mathrm{i}K')=0$. The latter property is equivalent to (3.2). The eigenfunction w will be called a Lamé–Wangerin function of the first kind. Note that we consider this eigenvalue problem for all real ν not just for $\nu\ge-\frac12$. Also note that the condition $q_0\ne0$ is not required in (3.4) although $q_0\ne0$ will hold if $\nu+\frac12$ is not a negative integer.

Similarly, we call h an eigenvalue of the second Lamé–Wangerin problem if (1.1) admits a nontrivial solution w on the interval $(\mathrm{i}K',2K+\mathrm{i}K')$ which close to $z=\mathrm{i}K'$ has the form (3.4) and satisfies $w(K+\mathrm{i}K')=0$. The latter property is equivalent to (3.3). The eigenfunction w will be called a Lamé–Wangerin function of the second kind.

If $\nu>-\frac32$ our eigenvalue problems are included in singular Sturm–Liouville theory (see also [15]) but this theory does not give us results for $\nu\le-\frac32$. We will treat these eigenvalue problems by a different method developed below.

We substitute

$$\eta=\mathrm{e}^{-2\mathrm{i}t} \tag{3.5}$$

in (2.5). We obtain the Fuchsian equation

$$\begin{aligned} k^2\eta(\eta-\eta_1)(\eta-\eta_2)&\left[\frac{\mathrm{d}^2w}{\mathrm{d}\eta^2}+\frac12\left(\frac1\eta+\frac{1}{\eta-\eta_1}+\frac{1}{\eta-\eta_2}\right)\frac{\mathrm{d}w}{\mathrm{d}\eta}\right]\\ &+\left(h-k^2\nu(\nu+1)\frac{(1+\eta)^2}{4\eta}\right)w=0, \end{aligned} \tag{3.6}$$

where

$$\eta_1 := \frac{1-k'}{1+k'} \in (0,1), \qquad \eta_2 := \frac{1+k'}{1-k'} \in (1,\infty).$$

The differential equation (3.6) has regular singularities at $\eta = 0, \eta_1, \eta_2, \infty$ with exponents $\{-\frac{1}{2}\nu, \frac{1}{2}(\nu+1)\}$, $\{0, \frac{1}{2}\}$, $\{0, \frac{1}{2}\}$, $\{-\frac{1}{2}\nu, \frac{1}{2}(\nu+1)\}$, respectively. If we combine (2.4) with (3.5) we obtain

$$\eta = \mathrm{e}^{-2\mathrm{i}(\frac{1}{2}\pi - \operatorname{am} z)} = (\operatorname{sn} z + \mathrm{i}\operatorname{cn} z)^{-2} = (\operatorname{sn} z - \mathrm{i}\operatorname{cn} z)^2. \tag{3.7}$$

Setting $z = u + \mathrm{i}K'$ for $0 < u < K$ this gives

$$\eta = \frac{1 - \operatorname{dn} u}{1 + \operatorname{dn} u}.$$

This establishes a bijective increasing map between $u \in (0, K)$ and $\eta \in (0, \eta_1)$. Taking into consideration the behavior of η close to $u = 0$ and $u = K$ we see that a Lamé–Wangerin function of the first kind expressed in the variable η is a solution of (3.6) on $(0, \eta_1)$ which close to $\eta = 0$ is of the form

$$w(\eta) = \eta^{(\nu+1)/2} \sum_{n=0}^{\infty} c_n \eta^n, \tag{3.8}$$

and which is analytic at $\eta = \eta_1$. This implies that the radius of convergence of the power series in (3.8) is $\geq \eta_2$. For the coefficients c_n we find the recursion

$$\begin{aligned} &\big(\beta_0^{(1)} - h\big)c_0 + \gamma_1 c_1 = 0, \\ &\alpha_n c_{n-1} + \big(\beta_n^{(1)} - h\big)c_n + \gamma_{n+1}c_{n+1} = 0, \qquad n \geq 1, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \alpha_n &= -\tfrac{1}{2}k^2(n+\nu)(2n-1), \\ \beta_n^{(1)} &= \tfrac{1}{2}k^2\nu(\nu+1) + \big(1 - \tfrac{1}{2}k^2\big)(2n+\nu+1)^2, \\ \gamma_n &= -\tfrac{1}{2}k^2(2n+2\nu+1)n. \end{aligned}$$

Note that the equations (3.9) for $n \geq 1$ agree with (2.7) when we set $\mu = \nu + 1$. The recursion (3.9) is given in [6, Section 15.6(15)].

Using Perron's rule, we see that h is an eigenvalue of the first Lamé–Wangerin problem if and only if (3.9) has a nontrivial solution $\{c_n\}_{n=0}^{\infty}$ which is either identically zero for large n or satisfies (2.8). Of course, a nontrivial solution $\{c_n\}_{n=0}^{\infty}$ of (3.9) can be identically zero for large n only if one of the numbers α_n is zero, that is, if ν is a negative integer.

Alternatively, we may expand a Lamé–Wangerin function of the first kind in the form

$$w(\eta) = \eta^{(\nu+1)/2}(\eta_2 - \eta)^{1/2} \sum_{n=0}^{\infty} a_n \eta^n \tag{3.10}$$

with the power series having radius $\geq \eta_2$. In order to find the recursion for the coefficients a_n we transform (3.6) by setting

$$w(\eta) = (\eta_2 - \eta)^{1/2} v(\eta)$$

to

$$k^2\eta(\eta-\eta_1)(\eta-\eta_2)\left[\frac{\mathrm{d}^2v}{\mathrm{d}\eta^2}+\frac{1}{2}\left(\frac{1}{\eta}+\frac{1}{\eta-\eta_1}+\frac{3}{\eta-\eta_2}\right)\frac{\mathrm{d}v}{\mathrm{d}\eta}\right] \\ +\left(h-k^2\nu(\nu+1)\frac{(1+\eta)^2}{4\eta}+\tfrac{1}{4}k^2(2\eta-\eta_1)\right)v=0. \tag{3.11}$$

We obtain the recursion

$$\begin{aligned}&\big(\epsilon_0^{(1)}-h\big)a_0+\delta_1a_1=0,\\ &\delta_na_{n-1}+\big(\epsilon_n^{(1)}-h\big)a_n+\delta_{n+1}a_{n+1}=0, \qquad n\ge 1,\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}\delta_n&=-\tfrac{1}{2}k^2n(2n+2\nu+1),\\ \epsilon_n^{(1)}&=\tfrac{1}{2}k^2\nu(\nu+1)+\big(1-\tfrac{1}{2}k^2\big)(2n+\nu+1)^2+\tfrac{1}{4}k^2\eta_1(4n+2\nu+3)\\ &=\tfrac{1}{2}k^2\nu(\nu+1)-k'\big(2n+\tfrac{3}{2}+\nu\big)+\big(1-\tfrac{1}{2}k^2\big)\big(\tfrac{1}{4}+\big(2n+\tfrac{3}{2}+\nu\big)^2\big).\end{aligned}$$

It is a pleasant surprise that, in contrast to (3.9), recursion (3.12) is of self-adjoint form. We take advantage of this observation and introduce a symmetric operator $S=S^{(1)}(\nu,k)$ in the Hilbert space $\ell^2(\mathbb{N}_0)$ with the standard inner product. The domain of definition of S is

$$D(S)=\left\{\{x_n\}_{n=0}^{\infty}\colon\ \sum_{n=0}^{\infty}n^4|x_n|^2<\infty\right\}$$

and S is defined on $D(S)$ by

$$\begin{aligned}&S(\{x_j\})_0=\epsilon_0^{(1)}x_0+\delta_1x_1,\\ &S(\{x_j\})_n=\delta_nx_{n-1}+\epsilon_n^{(1)}x_n+\delta_{n+1}x_{n+1}, \qquad n\ge 1.\end{aligned}$$

So S is represented by an infinite symmetric tridiagonal matrix.

Theorem 3.1. *Let $\nu\in\mathbb{R}$ and $k\in[0,1)$.*

(a) *$S^{(1)}(\nu,k)$ is a self-adjoint operator in $\ell^2(\mathbb{N}_0)$ with compact resolvent and bounded below.*

(b) *If $k\in(0,1)$ the eigenvalues of $S^{(1)}(\nu,k)$ agree with the eigenvalues of the first Lamé–Wangerin problem.*

(c) *If $k\in(0,1)$ the eigenvalues of $S^{(1)}(\nu,k)$ are simple.*

Proof. (a) We abbreviate $S=S^{(1)}(\nu,k)$, and write $S=A+B$ with $A=S^{(1)}(\nu,0)$. So A is represented by an infinite diagonal matrix with diagonal entries $(2n+\nu+1)^2$, $n\in\mathbb{N}_0$. It is clear that A is a positive semi-definite self-adjoint operator with compact resolvent. There are two constants $\lambda>0$ and $c\in(0,1)$ such that

$$\|Bx\|\le c\|(A+\lambda)x\| \qquad \text{for all } x\in D(S). \tag{3.13}$$

To prove this it is convenient to write $B=B_1+B_2+B_3$ where each B_i has a matrix representation consisting of only one nonzero "diagonal", and estimate $\|Bx\|\le\|B_1x\|+\|B_2x\|+\|B_3x\|$. We can reach $c<1$ because the factor of n^2 on the main diagonal of A is 4 while the factors of n^2 on the three diagonals of B are $-k^2$, $-2k^2$, $-k^2$, respectively. From (3.13) we obtain that

$T := B(A+\lambda)^{-1}$ is a bounded linear operator with operator norm $\|T\| \le c < 1$. Therefore, $1+T$ is invertible and

$$(S+\lambda)^{-1} = (A+\lambda+B)^{-1} = (A+\lambda)^{-1}(1+T)^{-1}.$$

This shows that $(S+\lambda)^{-1}$ is a compact operator. Since S is symmetric, we find that S is self-adjoint; compare [11, Chapter V, Theorem 4.3]. From (3.13) we also obtain that $S+\lambda$ is positive definite [11, Chapter V, Theorem 4.11]. Therefore, (a) follows.

(b) h is an eigenvalue of S if and only if the recursion (3.12) has a nontrivial solution $\{a_n\}_{n=0}^\infty$ with the property that $\sum\limits_{n=0}^{\infty} n^4|a_n|^2 < \infty$. By Perron's rule the latter property is equivalent to $a_n = 0$ for large n or $\{a_n\}$ is recessive as $n \to \infty$.

(c) If $k \in (0,1)$ the eigenvalues of S are simple because the corresponding eigenfunctions of the first Lamé–Wangerin problem are even with respect to $K + \mathrm{i}K'$. ■

Based on Theorem 3.1 we write the eigenvalues of the first Lamé–Wangerin problem with $k \in (0,1)$ in the form

$$H_0^{(1)}(\nu,k) < H_1^{(1)}(\nu,k) < H_2^{(1)}(\nu,k) < \cdots .$$

The Lamé–Wangerin function belonging to $H_m^{(1)}(\nu,k)$ will be denoted by $w_m^{(1)}(z,\nu,k)$. If a normalization is required it will be stated separately. We note that the corresponding eigenvectors $\{a_n\}_{n=0}^\infty$ of S when properly normalized form an orthonormal basis in the Hilbert space $\ell^2(\mathbb{N}_0)$.

The eigenvalues of $S^{(1)}(\nu,0)$ are $(2n+\nu+1)^2$ for $n \in \mathbb{N}_0$. If we arrange this sequence in increasing order repeated according to multiplicity we denote these eigenvalues by $H_m^{(1)}(\nu,0)$. Explicitly, they are given by the following lemma.

Lemma 3.2. *Let $p-1 < \nu \le p$ with $p \in \mathbb{Z}$. Then, for all $m \in \mathbb{N}_0$,*

$$H_m^{(1)}(\nu,0) = (2\ell+\nu+1)^2,$$

where

$$\ell = \begin{cases} m & \text{if } m+p \ge 0, \\ \frac{1}{2}(m-p) & \text{if } m+p < 0,\ m+p \text{ even}, \\ \frac{1}{2}(-m-p-1) & \text{if } m+p < 0,\ m+p \text{ odd}. \end{cases}$$

We will need continuity of the eigenvalues $H_m^{(1)}(\nu,k)$.

Theorem 3.3. *The function $(\nu,k) \mapsto H_m^{(1)}(\nu,k)$ is continuous on $\mathbb{R} \times [0,1)$ for every $m \in \mathbb{N}_0$.*

Proof. Set $S(\nu,k) = S^{(1)}(\nu,k)$, $A(\nu) = S(\nu,0)$ and $S(\nu,k) = A(\nu) + B(\nu,k)$. Let $\nu_0 > 0$ and $k_0 \in (0,1)$ be given, and set $\Omega := [-\nu_0,\nu_0] \times [0,k_0]$. Then we can find $\lambda > 0$ large enough and $c \in (0,1)$ such that (3.13) holds uniformly for $(\nu,k) \in \Omega$. It follows that $T(\nu,k) := B(\nu,k)(A(\nu)+\lambda)^{-1}$ is a bounded linear operator with operator norm $\|T(\nu,k)\| \le c$ for all $(\nu,k) \in \Omega$. As before, we have

$$(S(\nu,k)+\lambda)^{-1} = (A(\nu)+\lambda)^{-1}(1+T(\nu,k))^{-1}, \qquad (\nu,k) \in \Omega. \tag{3.14}$$

Suppose we have a sequence $(\nu_n,k_n) \in \Omega$ which converges to $\big(\hat\nu,\hat k\big)$ as $n \to \infty$. Then we can easily show using the definitions of A and T that

$$\big\|(A(\nu_n)+\lambda)^{-1} - (A(\hat\nu)+\lambda)^{-1}\big\| \to 0 \qquad \text{as } n \to \infty,$$

and

$$\left\|T(\nu_n,k_n)-T\big(\hat\nu,\hat k\big)\right\|\to 0 \qquad \text{as } n\to\infty.$$

Using (3.14) we then obtain that

$$\left\|(S(\nu_n,k_n)+\lambda)^{-1}-\big(S\big(\hat\nu,\hat k\big)+\lambda\big)^{-1}\right\|\to 0 \qquad \text{as } n\to\infty. \tag{3.15}$$

If K_n is a sequence of positive definite compact Hermitian operators converging to a positive definite compact Hermitian operator K with respect to the operator norm, then the mth largest eigenvalue of K (counted according to multiplicity) converges to the m largest eigenvalue of K as $n\to\infty$ for every $m\in\mathbb{N}_0$. This follows directly from the minimum-maximum-principle If we set $K_n=(S(\nu_n,k_n)+\lambda)^{-1}$, $K=\big(S\big(\hat\nu,\hat k\big)+\lambda\big)^{-1}$ and use (3.15) we obtain $H_m^{(1)}(\nu_n,k_n)\to H_m^{(1)}\big(\hat\nu,\hat k\big)$ as $n\to\infty$ for every $m\in\mathbb{N}_0$ as desired. ∎

A Lamé–Wangerin function of the second kind can be written in the form

$$w(\eta)=\eta^{(\nu+1)/2}(\eta_1-\eta)^{1/2}(\eta_2-\eta)^{1/2}\sum_{n=0}^{\infty}d_n\eta^n, \tag{3.16}$$

where the power series $\sum d_n\eta^n$ has radius $\geq \eta_2$. If we set

$$w(\eta)=(\eta-\eta_1)^{1/2}(\eta-\eta_2)^{1/2}v(\eta)$$

in (3.6), we obtain

$$k^2\eta(\eta-\eta_1)(\eta-\eta_2)\left[\frac{\mathrm{d}^2v}{\mathrm{d}\eta^2}+\frac12\left(\frac1\eta+\frac{3}{\eta-\eta_1}+\frac{3}{\eta-\eta_2}\right)\frac{\mathrm{d}v}{\mathrm{d}\eta}\right]$$
$$+\left(\tfrac32k^2\eta-1+\tfrac12k^2+h-k^2\nu(\nu+1)\frac{(1+\eta)^2}{4\eta}\right)v=0.$$

This gives the recursion

$$\begin{aligned}&\big(\beta_0^{(2)}-h\big)d_0+\gamma_1d_1=0,\\ &\alpha_{n+1}d_{n-1}+\big(\beta_n^{(2)}-h\big)d_n+\gamma_{n+1}d_{n+1}=0, \quad n\geq 1,\end{aligned} \tag{3.17}$$

where α_n, γ_n are as in (3.9) and

$$\beta_n^{(2)}=\tfrac12k^2\nu(\nu+1)+\big(1-\tfrac12k^2\big)(2n+\nu+2)^2.$$

Note that the equations (3.17) for $n\geq1$ agree with (2.9) when we set $\mu=\nu+2$. The recursion (3.17) is given in [6, Section 15.6(16)].

Alternatively, a Lamé–Wangerin function of the second kind can be written as

$$w(\eta)=\eta^{(\nu+1)/2}(\eta_1-\eta)^{1/2}\sum_{n=0}^{\infty}b_n\eta^n, \tag{3.18}$$

where the power series $\sum b_n\eta^n$ has radius $\geq\eta_2$. If we set

$$w(\eta)=(\eta_1-\eta)^{1/2}v(\eta)$$

in (3.6), we obtain (3.11) with η_1, η_2 interchanged. This gives the recursion

$$\begin{aligned}&\big(\epsilon_0^{(2)}-h\big)b_0+\delta_1b_1=0,\\ &\delta_nb_{n-1}+\big(\epsilon_n^{(2)}-h\big)b_n+\delta_{n+1}b_{n+1}=0, \qquad n\geq1,\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\delta_n &= -\tfrac{1}{2}k^2 n(2n+2\nu+1),\\ \epsilon_n^{(2)} &= \tfrac{1}{2}k^2\nu(\nu+1)+\big(1-\tfrac{1}{2}k^2\big)(2n+\nu+1)^2+\tfrac{1}{4}k^2\eta_2(4n+2\nu+3)\\ &= \tfrac{1}{2}k^2\nu(\nu+1)+k'\big(2n+\tfrac{3}{2}+\nu\big)+\big(1-\tfrac{1}{2}k^2\big)\big(\tfrac{1}{4}+\big(2n+\tfrac{3}{2}+\nu\big)^2\big).\end{aligned}$$

For the second Lamé–Wangerin problem we have results parallel to Theorem 3.1, Lemma 3.2 and Theorem 3.3. We denote the eigenvalues of the second Lamé–Wangerin problem by $H_m^{(2)}(\nu,k)$, and the corresponding Lamé–Wangerin eigenfunctions by $w_m^{(2)}(z,\nu,k)$.

4 Analytic continuation of Lamé–Wangerin functions

In the previous section Lamé–Wangerin functions were defined on the interval $(\mathrm{i}K', 2K+\mathrm{i}K')$. We analytically continue these functions to the strip $0\le \Im z < K'$ as follows. Using (3.7) and (3.8) a Lamé–Wangerin function $w^{(1)}$ of the first kind can be written as

$$w^{(1)}(z)=\mathrm{e}^{-\mathrm{i}(\nu+1)(\frac{1}{2}\pi-\operatorname{am} z)}\sum_{n=0}^{\infty} c_n(\operatorname{sn} z-\mathrm{i}\operatorname{cn} z)^{2n}. \tag{4.1}$$

Since the power series $\sum c_n\eta^n$ has radius larger than 1 and $|\eta|\le 1$ for $0\le\Im z<K'$, the expansion (4.1) converges in the strip $0\le \Im z<K'$.

If $0<\eta<\eta_1$ we have

$$\frac{1}{4}k^2\big(\eta^{1/2}+\eta^{-1/2}\big)^2-1=\frac{1}{4}k^2\eta^{-1}(\eta_1-\eta)(\eta_2-\eta). \tag{4.2}$$

If z is on the segment $(\mathrm{i}K', K+\mathrm{i}K')$ and η is given by (3.7) then (4.2) implies

$$\mathrm{i}\operatorname{dn} z=\frac{1}{2}k\eta^{-1/2}(\eta_1-\eta)^{1/2}(\eta_2-\eta)^{1/2}. \tag{4.3}$$

Therefore, (3.16) implies

$$w^{(2)}(z)=2\mathrm{i}k^{-1}\mathrm{e}^{-\mathrm{i}(\nu+2)(\frac{1}{2}\pi-\operatorname{am} z)}\operatorname{dn} z\sum_{n=0}^{\infty} d_n(\operatorname{sn} z-\mathrm{i}\operatorname{cn} z)^{2n}. \tag{4.4}$$

Again, this expansion is convergent in the strip $0\le\Im z<K'$.

In order to deal with expansions (3.10) and (3.18) we introduce the function

$$I_1(z):=(\operatorname{dn} z+\operatorname{cn} z)^{1/2} \tag{4.5}$$

also appearing in [8]. This function is analytic in the strip $-K'<\Im z<K'$ when the branch of the root is chosen as follows. The function $\operatorname{dn} z+\operatorname{cn} z$ does not assume negative values or zero in the rectangle $-2K<\Re z<2K$, $-K'<\Im z<K'$. We choose the principal branch of the root in (4.5) in this rectangle. We choose positive imaginary roots on the segments $(-2K,-2K+\mathrm{i}K')$ and $(2K-\mathrm{i}K', 2K)$. For other z, $I_1(z)$ is determined by $I_1(z+4K)=-I_1(z)$. A second function is defined by

$$I_2(z):=-I_1(z+2K)=-(\operatorname{dn} z-\operatorname{cn} z)^{1/2}.$$

For $0<\eta<\eta_1$ we have the identity

$$(1-\eta+q)^{1/2}+(1-\eta-q)^{1/2}=2^{1/2}(1-k')^{1/2}(\eta_2-\eta)^{1/2}, \tag{4.6}$$

where

$$q = k(\eta_1 - \eta)^{1/2}(\eta_2 - \eta)^{1/2}$$

and all roots denote positive roots of positive numbers. For z between $\mathrm{i}K'$ and $K + \mathrm{i}K'$ we have

$$\mathrm{i}\,\mathrm{cn}\, z = \frac{1}{2}\left(\eta^{-1/2} - \eta^{1/2}\right).$$

Therefore, it follows from (4.3) and (4.6) that the analytic continuation of $J_1 = \eta^{-1/4}(\eta_2 - \eta)^{1/2}$ to the strip $-K' < \Im z < K'$ is given by

$$(1 - k')^{1/2} J_1(z) = \mathrm{e}^{\mathrm{i}\frac{1}{4}\pi} I_1(z) + \mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi} I_2(z). \tag{4.7}$$

Therefore, the analytic continuation of the Lamé–Wangerin function $w^{(1)}$ of the first kind given by (3.10) to the strip $0 \le \Im z < K'$ is

$$w^{(1)}(z) = \mathrm{e}^{-\mathrm{i}(\nu+\frac{3}{2})(\frac{1}{2}\pi - \mathrm{am}\, z)} J_1(z) \sum_{n=0}^{\infty} a_n (\mathrm{sn}\, z - \mathrm{i}\,\mathrm{cn}\, z)^{2n}.$$

Similarly, for $0 < \eta < \eta_1$ we have

$$(1 - \eta + q)^{1/2} - (1 - \eta - q)^{1/2} = 2^{1/2}(1 + k')^{1/2}(\eta_1 - \eta)^{1/2}.$$

It follows that the analytic continuation of the function $J_2 = \eta^{-1/4}(\eta_1 - \eta)^{1/2}$ to the strip $-K' < \Im z < K'$ is given by

$$(1 + k')^{1/2} J_2(z) = \mathrm{e}^{\mathrm{i}\frac{1}{4}\pi} I_1(z) - \mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi} I_2(z). \tag{4.8}$$

If a Lamé–Wangerin function $w^{(2)}$ of the second kind is given by (3.18) then its analytic continuation is

$$w^{(2)}(z) = \mathrm{e}^{-\mathrm{i}(\nu+\frac{3}{2})(\frac{1}{2}\pi - \mathrm{am}\, z)} J_2(z) \sum_{n=0}^{\infty} b_n (\mathrm{sn}\, z - \mathrm{i}\,\mathrm{cn}\, z)^{2n}.$$

5 Comparison of eigenvalues

Every Lamé–Wangerin function is also a Floquet eigenfunction.

Lemma 5.1. *Let $\nu \in \mathbb{R}$ and $0 < k < 1$. A Lamé–Wangerin function $w^{(1)}(z)$ of the first kind satisfies*

$$w^{(1)}(z + 2K) = \mathrm{e}^{\mathrm{i}(\nu+1)\pi} w^{(1)}(z).$$

A Lamé–Wangerin function $w^{(2)}(z)$ of the second kind satisfies

$$w^{(2)}(z + 2K) = \mathrm{e}^{\mathrm{i}\nu\pi} w^{(2)}(z).$$

Proof. This follows from (4.1) and (4.4). ■

If $\mu + \nu$ or $\mu - \nu$ is an integer then the Floquet eigenvalues $h_m(\mu, \nu)$ can be expressed in terms of Lamé–Wangerin eigenvalues $H_m^{(j)}(\nu)$. The properties (2.2) show that it is sufficient to consider the case $\mu = \nu + 1$. Then we have the following result.

Theorem 5.2. *Let $0 < k < 1$, $\nu \in \mathbb{R}$, $p \in \mathbb{N}_0$ and $p - 1 < |\nu| \le p$.*

(*a*) *If* $\nu \ge 0$ *then*

$$\begin{aligned} &h_m(\nu+1,\nu) = H_m^{(2)}(-\nu-1), \qquad m = 0,1,\dots,p-1,\\ &h_{p+2i+1}(\nu+1,\nu) = H_{p+i}^{(2)}(-\nu-1), \qquad i \ge 0,\\ &h_{p+2i}(\nu+1,\nu) = H_i^{(1)}(\nu), \qquad i \ge 0. \end{aligned}$$

(*b*) *If* $\nu < 0$ *then*

$$\begin{aligned} &h_m(\nu+1,\nu) = H_m^{(1)}(\nu), \qquad m = 0,1,\dots,p-1,\\ &h_{p+2i+1}(\nu+1,\nu) = H_{p+i}^{(1)}(\nu), \qquad i \ge 0,\\ &h_{p+2i}(\nu+1,\nu) = H_i^{(2)}(-\nu-1), \qquad i \ge 0. \end{aligned}$$

(*c*) *If* ν *is an integer then*

$$h_0(\nu+1,\nu) < h_1(\nu+1,\nu) < \dots < h_p(\nu+1,\nu),$$

and, for $i \ge 0$,

$$h_{p+2i}(\nu+1,\nu) = h_{p+2i+1}(\nu+1,\nu) < h_{p+2i+2}(\nu+1,\nu) = h_{p+2i+3}(\nu+1,\nu) < \cdots .$$

Proof. (a), (b) Suppose first that ν is not an integer. Then the eigenvalues $f_m(k) := h_m(\nu+1,\nu,k)$ form a strictly increasing sequence. By Lemma 5.1, each eigenvalue $g_m(k) := H_m^{(1)}(\nu,k)$ and $G_m(k) := H_m^{(2)}(-\nu-1,k)$ is among the f-eigenvalues. This is also true for $k=0$. The sequence $\{f_m(0)\}_{m=0}^\infty$ is given by the sequence $\left\{(2n+\nu+1)^2\right\}_{n\in\mathbb{Z}}$ when arranged in increasing order, $\{g_m(0)\}$ is given by $\{(2n+\nu+1)^2\}_{n=0}^\infty$ in increasing order and $\{G_m(0)\}$ is given by $\left\{(2n+\nu+1)^2\right\}_{n=-\infty}^{-1}$ in increasing order. Because of continuity of the functions f_m, g_m, G_m (Theorems 2.1 and 3.3) the order of these eigenvalues is the same for all $k \in [0,1)$. An analysis of the order at $k=0$ proves (a) and (b) for noninteger ν. The result extends to integer ν by continuity.

(c) Let ν be an integer. Then we apply (a) or (b) to $\nu+\epsilon$ in place of ν and take limits $\epsilon \to 0^\pm$. This proves (c). ■

We now compare the eigenvalues $H_m^{(j)}(\nu)$ with $H_m^{(j)}(-\nu-1)$.

Theorem 5.3. *Let* $p \in \mathbb{N}_0$, $0 < k < 1$. *Let either* $H = H^{(1)}$ *or* $H = H^{(2)}$.

(*a*) *If* $-p-\frac{3}{2} < \nu < -p-\frac{1}{2}$ *then*

$$H_p(\nu) < H_0(-\nu-1) < H_{p+1}(\nu) < H_1(-\nu-1) < H_{p+2}(\nu) < \cdots .$$

(*b*) *If* $\nu = -p-\frac{1}{2}$ *then*

$$H_{p-1}(\nu) < H_0(-\nu-1) = H_p(\nu) < H_1(-\nu-1) = H_{p+1}(\nu) < \cdots .$$

Proof. We consider the eigenvalues $H_m = H_m^{(1)}$. The proof for $H_m = H_m^{(2)}$ is similar.

(a) Let $-p-\frac{3}{2} < \nu < -p-\frac{1}{2}$. The eigenvalues $H_m(\nu)$, $m \ge 0$, are pairwise distinct, and the eigenvalues $H_\ell(-\nu-1)$, $\ell \ge 0$, are pairwise distinct. The eigenvalues $H_m(\nu)$ are also distinct from $H_\ell(-\nu-1)$ because the corresponding eigenfunctions are linearly independent. Therefore, using continuity of the functions $k \mapsto H_m(\nu,k)$ (Theorem 3.3) the order of the eigenvalues $H_m(\nu,k)$, $H_\ell(-\nu-1,k)$ must be the same for all $k \in [0,1)$. The sequence $\{H_m(\nu,0)\}_{m=0}^\infty$ agrees with $\left\{(2n+\nu+1)^2\right\}_{n=0}^\infty$ after the latter sequence is arranged in increasing order. Similarly, the sequence $\{H_m(-\nu-1,0)\}_{m=0}^\infty$ agrees with $\left\{(2n-\nu)^2\right\}_{n=0}^\infty$ arranged in increasing order. Analysis of this order at $k=0$ implies (a).

(b) If $p=0$ then $\nu=-\frac{1}{2}$ and (b) is trivially true because $\nu=-\nu-1$. For $p\ge 1$ statement (b) follows from continuity of the functions $\nu\mapsto H(\nu,k)$ and taking one-sided limits as $\nu\to -p-\frac{1}{2}$ in (a). ∎

We now compare the eigenvalues $H_m^{(1)}(\nu)$ with $H_m^{(2)}(\nu)$.

Theorem 5.4. *Let $\nu\in\mathbb{R}$, $0<k<1$ and $H_m^{(j)}:=H_m^{(j)}(\nu,k)$.*

(*a*) *If $\nu>-\frac{3}{2}$ then*

$$H_0^{(1)}<H_0^{(2)}<H_1^{(1)}<H_1^{(2)}<H_2^{(1)}<H_2^{(2)}<\cdots .$$

(*b*) *If $-p-\frac{3}{2}<\nu<-p-\frac{1}{2}$ with $p\in\mathbb{N}$ then*

$$\begin{Bmatrix} H_0^{(1)} \\ H_0^{(2)} \end{Bmatrix}<\cdots<\begin{Bmatrix} H_{p-1}^{(1)} \\ H_{p-1}^{(2)} \end{Bmatrix}<H_p^{(1)}<H_p^{(2)}<H_{p+1}^{(1)}<H_{p+1}^{(2)}<\cdots ,$$

where, for $m=0,1,\dots,p-1$,

$$H_m^{(1)}<H_m^{(2)}\quad \text{if } m+p \text{ is even},\qquad H_m^{(1)}>H_m^{(2)}\quad \text{if } m+p \text{ is odd.}$$

(*c*) *If $\nu=-p-\frac{1}{2}$ with $p\in\mathbb{N}$ then*

$$H_0^{(1)}=H_0^{(2)}<H_1^{(1)}=H_2^{(2)}<\cdots<H_{p-1}^{(1)}=H_{p-1}^{(2)}<H_p^{(1)}<H_p^{(2)}<\cdots .$$

Proof. (a) Let $\nu>-\frac{3}{2}$. Then the eigenfunctions of the two Lamé–Wangerin problems are constant multiples of solution (3.4) with $q_0\neq 0$. Therefore, the eigenvalues of the two problems are mutually distinct. By continuity of the functions $k\mapsto H_m^{(j)}(\nu,k)$ (Theorem 3.3) the order of the eigenvalues $H_m^{(j)}$ must be the same for all $k\in[0,1)$. Now

$$H_m^{(1)}(\nu,0)=(2m+\nu+1)^2,\qquad H_m^{(2)}(\nu,0)=(2m+\nu+2)^2,$$

which implies (a).

(b) Let $-p-\frac{3}{2}<\nu<-p-\frac{1}{2}$ with $p\in\mathbb{N}$. Again, the eigenvalues of the two Lamé–Wangerin problems are mutually distinct, and the order of these eigenvalues must be the same for all $k\in[0,1)$. The sequence $\left\{H_m^{(j)}(\nu,0)\right\}_{m=0}^\infty$ is the same as $\{(2n+\nu+j)^2\}_{n=0}^\infty$ but the latter one has to be ordered increasingly. An analysis of the order leads to the arrangement stated in (b).

(c) Let $\nu=-p-\frac{1}{2}$ with $p\in\mathbb{N}$. Continuity of the functions $\nu\mapsto H_m^{(j)}(\nu,k)$ and part (b) show that $H_m^{(1)}=H_m^{(2)}$ for $m=0,1,\dots,p-1$. We know from Theorem 5.3(b) that

$$H_{m+p}^{(j)}(\nu)=H_m^{(j)}(-\nu-1),\qquad m\ge 0.$$

Since $-\nu-1>-\frac{3}{2}$ the rest of statement (c) follows from part (a). ∎

6 Algebraic Lamé functions

If $\nu+\frac12$ is a nonzero integer then Lamé's differential equation (1.1) has solutions in finite terms which are usually called algebraic Lamé functions. These solutions were investigated in [4, 7, 8, 13]. We obtain these functions as follows.

Let $\nu=-p-\frac12$ with $p\in\mathbb{N}$. For $j=1,2$ we introduce the symmetric tridiagonal p by p matrices

$$S_p^{(j)}=\begin{pmatrix} \epsilon_0^{(j)} & \delta_1 & 0 & & \\ \delta_1 & \epsilon_1^{(j)} & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \epsilon_{p-2}^{(j)} & \delta_{p-1} \\ & & 0 & \delta_{p-1} & \epsilon_{p-1}^{(j)} \end{pmatrix},$$

where

$$\begin{aligned}\epsilon_n^{(j)} &= \tfrac12 k^2\bigl(p^2-\tfrac14\bigr)+(-1)^j k'(2n+1-p)+\bigl(1-\tfrac12 k^2\bigr)\bigl(\tfrac14+(2n+1-p)^2\bigr),\\ \delta_n &= k^2 n(p-n).\end{aligned}$$

The coefficient δ_p vanishes in (3.12), (3.19). Therefore, if $(a_0,a_1,\dots,a_{p-1})^t$ is an eigenvector of $S_p^{(j)}$ and $a_n:=0$ for $n\ge p$ then (3.10), (3.18) are Lamé–Wangerin functions of the first and second kind, respectively.

We note that $S_p^{(1)}$ is the mirror image of $S_p^{(2)}$ with respect to the anti-diagonal, that is, we have

$$\epsilon_n^{(1)}=\epsilon_{p-1-n}^{(2)},\qquad \delta_n=\delta_{p-n}.$$

It follows that $S_p^{(1)}$ and $S_p^{(2)}$ have the same eigenvalues and the corresponding eigenvectors are inverse to each other, that is, if $(a_0,a_1,\dots,a_{p-1})^t$ is an eigenvector for $S_p^{(1)}$ then $(a_{p-1},a_{p-2},\dots,a_0)^t$ is an eigenvector for $S_p^{(2)}$ belonging to the same eigenvalue. According to Theorem 5.4(c), the common eigenvalues of $S_p^{(j)}$ are

$$H_m^{(1)}(-p-\tfrac12)=H_m^{(2)}(-p-\tfrac12),\qquad m=0,1,\dots,p-1.$$

If $(a_0,a_1,\dots,a_{p-1})^t$ is a (real) eigenvector of $S_p^{(1)}$ then

$$w^{(1)}=\eta^{-\frac12 p+\frac14}(\eta_2-\eta)^{1/2}\sum_{n=0}^{p-1}a_n\eta^n, \tag{6.1}$$

$$w^{(2)}=\eta^{-\frac12 p+\frac14}(\eta_1-\eta)^{1/2}\sum_{n=0}^{p-1}a_{p-n-1}\eta^n \tag{6.2}$$

are solutions of (3.11). These are algebraic Lamé functions expressed in the variable η. We note that the functions $w^{(1)}$ and $w^{(2)}$ are essentially Heun polynomials. For if we set $w=\eta^{-\frac12 p+\frac14}(\eta_j-\eta)^{1/2}v(s)$ and $\eta=\eta_1 s$, then we obtain the Heun equation for $v(s)$ and $\sum_{n=0}^{p-1}a_n(\eta_1 s)^n$ is a Heun polynomial.

If we substitute (3.7) in (6.1), (6.2) and use the functions $J_1(z)$, $J_2(z)$ defined in (4.7), (4.8) we obtain

$$w^{(1)}(z) = J_1(z)\sum_{n=0}^{p-1} a_n(\operatorname{sn} z - \mathrm{i}\operatorname{cn} z)^{2n-p+1},$$
$$w^{(2)}(z) = J_2(z)\sum_{n=0}^{p-1} a_n(\operatorname{sn} z + \mathrm{i}\operatorname{cn} z)^{2n-p+1}.$$

We know from Lemma 5.1 that

$$w^{(1)}(z+2K) = \mathrm{i}(-1)^p w^{(1)}(z), \qquad w^{(2)}(z+2K) = \mathrm{i}(-1)^{p+1} w^{(2)}(z).$$

Moreover, we have

$$(1+k')^{1/2}\overline{w^{(1)}(\bar z)} = -\mathrm{i}(1-k')^{1/2} w^{(2)}(z),$$

and, for $x \in \mathbb{R}$,

$$\overline{w^{(1)}(x)} = w^{(1)}(2K-x), \qquad \overline{w^{(2)}(x)} = -w^{(2)}(2K-x),$$

which shows that the real part of $w^{(1)}(x)$ is a function even with respect to $x = K$ while the imaginary part of $w^{(1)}(x)$ is odd with respect to $x = K$.

One should notice that if $\nu = -p - \frac{1}{2}$ and h is an eigenvalue of the matrix $S_p^{(j)}$ then all (nontrivial) solutions of Lamé's equation qualify as "algebraic Lamé functions". We picked the basis of two solutions, one even and one odd with respect to $z = K + \mathrm{i}K'$. Ince [8] considered the basis of even and odd solutions (with respect to $z = 0$) while Erdélyi [4] has the basis of even or odd solutions with respect to $z = K$.

In the simplest case $\nu = -\frac{3}{2}$ we have

$$H_0^{(1)}\big(-\tfrac{3}{2}\big) = H_0^{(2)}\big(-\tfrac{3}{2}\big) = \tfrac{1}{4}\big(1+k^2\big)$$

and

$$w_0^{(1)}(z) = J_1(z), \qquad w_0^{(2)}(z) = J_2(z).$$

If $\nu = -\frac{5}{2}$ then

$$H_0^{(1)}\big(-\tfrac{5}{2}\big) = H_0^{(2)}\big(-\tfrac{5}{2}\big) = \tfrac{5}{4}\big(1+k^2\big) - \big(1-k^2+k^4\big)^{1/2},$$
$$H_1^{(1)}\big(-\tfrac{5}{2}\big) = H_1^{(2)}\big(-\tfrac{5}{2}\big) = \tfrac{5}{4}\big(1+k^2\big) + \big(1-k^2+k^4\big)^{1/2}.$$

If we choose $a_0 = -k^2$, $a_1 = \frac{3}{4}k^2 + \frac{9}{4} - \frac{1}{2}k^2\eta_1 - H_m^{(1)}\big(-\frac{5}{2}\big)$, $m = 0, 1$, then

$$w_m^{(1)}(z) = J_1(z)(a_0(\operatorname{sn} z + \mathrm{i}\operatorname{cn} z) + a_1(\operatorname{sn} z - \mathrm{i}\operatorname{cn} z)),$$
$$w_m^{(2)}(z) = J_2(z)(a_0(\operatorname{sn} z - \mathrm{i}\operatorname{cn} z) + a_1(\operatorname{sn} z + \mathrm{i}\operatorname{cn} z)).$$

7 Lamé polynomials

Let $\nu = -p - 1$ with $p \in \mathbb{N}_0$. It is well-known [1, Chapter IX] that there are $2p + 1$ distinct values of h for which (1.1) admits nontrivial solutions which are polynomials in $\operatorname{cn} z$, $\operatorname{sn} z$, $\operatorname{dn} z$. In our notation these values of h are

$$H_m^{(1)}(-p-1), \qquad m = 0, 1, \ldots, p \tag{7.1}$$

and

$$H_m^{(2)}(-p-1), \qquad m = 0, 1, \ldots, p-1. \tag{7.2}$$

Since $\alpha_{p+1} = 0$ in (3.9), the numbers (7.1) are the eigenvalues of the $p+1$ by $p+1$ tridiagonal matrix

$$T_{p+1}^{(1)} = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & & \\ \alpha_1 & \beta_1 & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \beta_{p-1} & \gamma_p \\ & & 0 & \alpha_p & \beta_p \end{pmatrix},$$

where

$$\begin{aligned} \alpha_n &= \tfrac{1}{2}k^2(p+1-n)(2n-1), \\ \beta_n &= \tfrac{1}{2}k^2 p(p+1) + \left(1 - \tfrac{1}{2}k^2\right)(2n-p)^2, \\ \gamma_n &= \tfrac{1}{2}k^2(2p+1-2n)n. \end{aligned}$$

If $(c_0, c_1, \ldots, c_p)^t$ is an eigenvector of $T_{p+1}^{(1)}$ then

$$w = \eta^{-p/2} \sum_{n=0}^{p} c_n \eta^n$$

is a solution of (3.6). After substituting (3.7) we obtain

$$w = \sum_{n=0}^{p} c_n (\operatorname{sn} z - \mathrm{i} \operatorname{cn} z)^{2n-p}.$$

Indeed, since $(\operatorname{sn} z - \mathrm{i} \operatorname{cn} z)^{-1} = \operatorname{sn} z + \mathrm{i} \operatorname{cn} z$, these solutions are polynomials in $\operatorname{cn} z$, $\operatorname{sn} z$. The matrix $T_{p+1}^{(1)}$ has the symmetries

$$\alpha_n = \gamma_{p-n+1}, \qquad \beta_n = \beta_{p-n}.$$

Therefore, the space of symmetric vectors $\{c_n\}_{n=0}^{p}$ ($c_n = c_{p-n}$, $n = 0, 1, \ldots, p$), as well as the space of antisymmetric vectors is invariant under $T_{p+1}^{(1)}$. Thus eigenvectors of $T_{p+1}^{(1)}$ will lie in one of these invariant subspaces.

If p is even we find $\frac{1}{2}p + 1$ Lamé polynomials of the form $P\left(\operatorname{sn}^2 z\right)$ where P is a polynomial of degree $\frac{1}{2}p$ if we use symmetric eigenvectors, and $\frac{1}{2}p$ Lamé polynomials of the form $\operatorname{cn} z \operatorname{sn} z P\left(\operatorname{sn}^2 z\right)$ where P is a polynomial of degree $\frac{1}{2}p - 1$ if we use antisymmetric eigenvectors. If p is odd we find $\frac{1}{2}(p+1)$ Lamé polynomials of the form $\operatorname{sn} z P\left(\operatorname{sn}^2 z\right)$ where P is a polynomial of degree $\frac{1}{2}(p-1)$ if we use symmetric eigenvectors, and $\frac{1}{2}(p+1)$ Lamé polynomials of the form $\operatorname{cn} z P\left(\operatorname{sn}^2 z\right)$ where P is a polynomial of degree $\frac{1}{2}(p-1)$ if we use antisymmetric eigenvectors.

Similarly, Lamé–Wangerin functions of the second kind belonging to the eigenvalues (7.2) are Lamé polynomials that have the factor $\operatorname{dn} z$.

8 Zeros of Lamé–Wangerin functions

We first determine the number of zeros of Lamé–Wangerin functions $w_m^{(j)}(z)$ in the open interval $(\mathrm{i}K', K+\mathrm{i}K')$.

Theorem 8.1. *Let $j=1,2$, $m\in\mathbb{N}_0$, $\nu\in\mathbb{R}$, $k\in(0,1)$.*

(*a*) *If $\nu>-\frac{3}{2}$ then $w_m^{(j)}$ has exactly m zeros in $(\mathrm{i}K', K+\mathrm{i}K')$.*

(*b*) *If $-p-\frac{3}{2}<\nu\le -p-\frac{1}{2}$, $p\in\mathbb{N}$, then $w_m^{(j)}$ has exactly $\max\{0,m-p\}$ zeros in $(\mathrm{i}K', K+\mathrm{i}K')$.*

Proof. Consider $j=1$. The proof for $j=2$ similar. Let P be the set of all real numbers different from $-p-\frac{1}{2}$ for all $p\in\mathbb{N}$. For $h\in\mathbb{R}$, $\nu\in P$, let $w(z,h,\nu)$ be the solution of (1.1) given locally at $z=\mathrm{i}K'$ by (3.4) with $q_0=1$. Then one can show that $(z-\mathrm{i}K')^{-\nu-1}w(z,h,\nu)$ is continuous for z in $[\mathrm{i}K', K+\mathrm{i}K']$, $h\in\mathbb{R}$, $\nu\in P$ (k fixed). If we set $w_m(z,\nu)=w\big(z,H_m^{(1)}(\nu,k),\nu\big)$ then $(z-\mathrm{i}K')^{-\nu-1}w_m(z,\nu)$ is continuous for $z\in[\mathrm{i}K', K+\mathrm{i}K']$ and $\nu\in P$. This implies that the number of zeros of $w_m(\cdot,\nu)$ in $(\mathrm{i}K', K+\mathrm{i}K')$ is finite and it is locally constant as a function of ν.

(a) follows by considering $\nu=0$:

$$w_m^{(1)}(z,0,k)=(-1)^m\sin\left((2m+1)\frac{\pi}{2K}(z-\mathrm{i}K')\right). \tag{8.1}$$

(b) Suppose $-p-\frac{3}{2}<\nu\le -p-\frac{1}{2}$ with $p\in\mathbb{N}$. Let $m=0,1,\dots,p$. By Theorem 5.3(a), we have $H_m(\nu)\le H_0(-\nu-1)$. By (a), $w_0(\cdot,-\nu-1)$ has no zeros in $(\mathrm{i}K', K+\mathrm{i}K')$. Therefore, by Sturm comparison, $w_m(\cdot,\nu)$ also has no zeros in this interval.

Now consider $m>p$. If $\nu=-p-\frac{1}{2}$ then, by Theorem 5.3(b), $H_m(\nu)=H_{m-p}(-\nu-1)$. Therefore, by (a), $w_m(\cdot,\nu)$ has $m-p$ zeros in $(\mathrm{i}K', K+\mathrm{i}K')$. If $-p-\frac{3}{2}<\nu<p-\frac{1}{2}$ then, by Theorem 5.3(a), $H_m(\nu)<H_{m-p}(-\nu-1)$. Therefore, $w_m(\cdot,\nu)$ can have at most $m-p$ zeros in $(\mathrm{i}K', K+\mathrm{i}K')$. If $\nu=-p-\frac{1}{2}$ we just showed that there are $m-p$ zeros. By continuity, there are exactly $m-p$ zeros. For the latter step Lamé–Wangerin functions should be normalized by the initial conditions $w(K+\mathrm{i}K')=1$, $w'(K+\mathrm{i}K')=0$. ■

Now we look for zeros of Lamé–Wangerin functions in the strip $0\le\Im z<K'$.

Lemma 8.2. *A Lamé–Wangerin function which is not a Lamé polynomial has no zeros on the real axis.*

Proof. If μ is not an integer then a nontrivial Floquet solution $w(z)$, $z\in\mathbb{R}$, of (1.1) with $w(z+2K)=\mathrm{e}^{\mathrm{i}\mu\pi}w(z)$ does not have zeros on the real axis. This is because the conjugate of $w(z)$ is a Floquet solution with conjugate multiplier $\mathrm{e}^{-\mathrm{i}\mu\pi}$, and $\mathrm{e}^{\mathrm{i}\mu\pi}$, $\mathrm{e}^{-\mathrm{i}\mu\pi}$ are distinct. So $w(z)$ and its conjugate function are linearly independent. It follows from Lemma 5.1 that Lamé–Wangerin functions have no zeros on the real axis if ν is not an integer.

Suppose that ν is an integer, and $w(z)$ is a Lamé–Wangerin function belonging to the eigenvalue $H_m^{(1)}(\nu)$. Suppose that $w(z_0)=0$. with $z_0\in\mathbb{R}$. Using (3.8) and the substitution (2.4) we have

$$w(t)=\sum_{n=0}^{\infty}c_n\mathrm{e}^{-\mathrm{i}t(2n+\nu+1)}$$

and this function has a zero at $t_0\in\mathbb{R}$. The coefficients c_n are real so the functions

$$\Re w(t)=\sum_{n=0}^{\infty}c_n\cos(2n+\nu+1)t,\qquad t\in\mathbb{R} \tag{8.2}$$

$$\Im w(t)=-\sum_{n=0}^{\infty}c_n\sin(2n+\nu+1)t,\qquad t\in\mathbb{R} \tag{8.3}$$

both vanish at $t = t_0$. The functions (8.2), (8.3) are both solutions of the differential equation (2.5) with the same values for h and ν. Since they have a common zero these solutions must be linearly dependent. Now $\Re w(t)$ is an even function and $\Im w(t)$ is odd. So one of the functions $\Re w(t)$, $\Im w(t)$ must vanish identically. This implies that $c_n = 0$ for large enough n and so $w(z)$ is a Lamé polynomial. The proof is similar for Lamé–Wangerin function of the second kind. ∎

According to (3.10) we write a Lamé–Wangerin function of the first kind as

$$w_m^{(1)} = \eta^{(\nu+1)/2}(\eta_2 - \eta)^{1/2} v_m^{(1)}(\eta, \nu, k),$$

where

$$v_m^{(1)}(\eta, \nu, k) = \sum_{n=0}^{\infty} a_n \eta^n$$

is given by a power series with radius $\geq \eta_2 > 1$. Similarly, we write a Lamé–Wangerin function of the second kind as

$$w_m^{(2)} = \eta^{(\nu+1)/2}(\eta_1 - \eta)^{1/2} v_m^{(2)}(\eta, \nu, k).$$

Theorem 8.3. *Let $m \in \mathbb{N}_0$, $\nu \in \mathbb{R}$, $k \in (0, 1)$.*

(a) *Suppose that $-m - \nu \notin \mathbb{N}$, and choose $\ell \in \mathbb{N}_0$ such that $H_m^{(1)}(\nu, 0) = (2\ell + \nu + 1)^2$; see Lemma 3.2. Then $v_m^{(1)}(\cdot, \nu, k)$ has exactly ℓ zeros in the unit disk $|\eta| < 1$ counted by multiplicity.*

(b) *Suppose that $-m - \nu - 1 \notin \mathbb{N}$, and choose $\ell \in \mathbb{N}_0$ such that $H^{(2)}(\nu, 0) = (2\ell + \nu + 2)^2$. Then $v_m^{(2)}(\cdot, \nu, k)$ has exactly ℓ zeros in the unit disk $|\eta| < 1$ counted by multiplicity.*

Proof. We prove only (a). The proof of (b) is similar. We normalize the Lamé–Wangerin functions $w_m(z)$ of the first kind by setting $w_m(K + \mathrm{i}K') = 1$. Then $w_m(z, \nu, k)$ is the solution of (1.1) with $h = H_m^{(1)}(\nu, k)$ determined by the initial conditions $w(K+\mathrm{i}K') = 1$, $w'(K+\mathrm{i}K') = 0$. By continuous parameter dependence of solutions of initial value problems of linear differential equations, and using Theorem 3.3, we obtain that $w_m(z, \nu, k)$ is a continuous function of (z, ν, k) for $z \in \mathbb{R}$, $\nu \in \mathbb{R}$, $k \in (0, 1)$. Since $|\eta| = 1$ is in correspondence with $z \in [0, 2K)$, we see that $v_m(\eta, \nu, k)$ is a continuous function of $|\eta| = 1$, $\nu \in \mathbb{R}$, $k \in (0, 1)$. We want to apply Rouché's theorem to the homotopy $s \mapsto v_m(\eta, s\nu, k)$ for $s \in [0, 1]$. If $v_m(\eta, s\nu, k) \neq 0$ on the unit circle $|\eta| = 1$ for all $s \in [0, 1]$, then $v_m(\cdot, s\nu, k)$ has the same number of zeros in $|\eta| < 1$ for $s \in [0, 1]$.

Suppose that $\nu > -m - 1$. By Lemma 8.2, the function $v_m(\eta, s\nu, k)$ has no zeros on the unit circle $|\eta| = 1$ for $0 \leq s \leq 1$ and so the number of zeros of $v_m(\cdot, \nu, k)$ in the open unit disk agrees with that of $v_m(\cdot, 0, k)$. It follows from (8.1) that the number of zeros of $v_m(\cdot, \nu, k)$ in the open unit disk is equal to m. Under our assumption on (ν, m) we have $\ell = m$, so we obtain statement (a) for $\nu > -m - 1$.

Now we assume that $-p - 1 < \nu < -p$ with $p \in \mathbb{N}$ and $m < p$. We use similar homotopies to show that the number of zeros of $v_m(\cdot, \nu, k)$ may depend on p and m but not on $\nu, k \in (0, 1)$. So we consider $\nu = -p - \frac{1}{2}$. Then w is an algebraic Lamé function and $v_m(\eta) = \sum\limits_{n=0}^{p-1} a_n \eta^n$ is a polynomial. Let $k_n \in (0, 1)$ be a sequence converging to 0. Since the vector $(a_0, a_1, \dots, a_{p-1})^t$ is an eigenvector of the matrix $S_p^{(1)}$ from Section 6 it is easy to see that when properly normalized the eigenvectors belonging to k_n converge to the vector $(a_0, \dots, a_{p-1})^t$ with all components equal to 0 except $a_\ell = 1$. Therefore, under the new normalization $v_m\big(\eta, -p - \frac{1}{2}, k_n\big)$ converges uniformly to η^ℓ as $n \to 0^+$. By Rouché's theorem, we obtain the desired statement.

This completes the proof. ∎

Using the map (3.7) the unit disk $|\eta| < 1$ can be related to a domain in the z-plane. Consider the rectangle

$$Q = \{z \in \mathbb{C}\colon 0 < \Re z < 2K,\, 0 < \Im z < K'\}.$$

The function $z \mapsto \eta$ is a conformal map from Q onto the unit disk $|\eta| < 1$ with a branch cut along the interval $(-1, \eta_1)$. If z starts at $z = 0$ and moves clockwise around the boundary of Q, then η starts at $\eta = -1$ and moves in the mathematically positive direction along the unit circle returning to $\eta = -1$ when $z = 2K$. Then η moves from $\eta = -1$ to $\eta = 0$ when z reaches $z = \mathrm{i}K'$. Then η moves to η_1 when $z = K + \mathrm{i}K'$ and returns to $\eta = 0$, then to $\eta = -1$. It follows that the set

$$\tilde{Q} = \{z\colon 0 \le \Re z \le K,\, 0 < \Im z \le K'\} \cup \{z\colon K < \Re z < 2K,\, 0 < \Im z < K'\}$$

is mapped bijectively onto the open unit disk $|\eta| < 1$.

Theorem 8.3(a) can be extended to include Lamé polynomials. If $\nu = -m-1, -m-2, \dots$ then let ℓ_1 be the smallest nonnegative integer n satisfying $H_m^{(1)}(\nu, 0) = (2n + \nu + 1)^2$ and ℓ_2 the largest such integer. Then v_m has ℓ_1 zeros in the open unit disk $|\eta| < 1$ and ℓ_2 zeros in the closed unit disk $|\eta| \le 1$. This follows from the known location of zeros of Lamé polynomials [1, Section 9.4]. Similarly, Theorem 8.3(b) can be extended.

9 The limit $k \to 0$ of Lamé–Wangerin functions

Substituting $u = \frac{2K}{\pi} s$ in (3.1), we obtain the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}s^2} + \frac{4K^2}{\pi^2}\left(h - \nu(\nu+1)\,\mathrm{sn}^{-2}\left(\frac{2K}{\pi}s\right)\right) w = 0, \qquad 0 < s < \pi. \tag{9.1}$$

In (9.1) we set $h = H_m^{(1)}(\nu, k)$ and take $w = w_m^{(1)}(s, \nu, k)$ as the corresponding Lamé–Wangerin eigenfunction normalized by the initial condition

$$w\left(\frac{\pi}{2}\right) = 1, \qquad \frac{\mathrm{d}w}{\mathrm{d}s}\left(\frac{\pi}{2}\right) = 0.$$

By Theorem 3.3, $H_m^{(1)}(\nu, k) \to H_m^{(1)}(\nu, 0) = (2\ell + \nu + 1)^2$ as $k \to 0^+$, where $\ell \in \mathbb{N}_0$ is chosen according to Lemma 3.2. As $k \to 0^+$ we see that $w_m^{(1)}(s, \nu, k)$ converges to the solution $W_m^{(1)}(s, \nu)$ of the differential equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}s^2} + \left((2\ell + \nu + 1)^2 - \frac{\nu(\nu+1)}{\sin^2 s}\right) W = 0 \tag{9.2}$$

satisfying the initial conditions

$$W\left(\frac{\pi}{2}\right) = 1, \qquad \frac{\mathrm{d}W}{\mathrm{d}s}\left(\frac{\pi}{2}\right) = 0.$$

The convergence

$$w_m^{(1)}(s, \nu, k) \to W_m^{(1)}(s, \nu) \qquad \text{as } k \to 0^+$$

is uniform on compact subintervals of $(0, \pi)$. Differential equation (9.2) appears in the theory of Gegenbauer polynomials [20, equation (4.7.11)]. We find that

$$W_m^{(1)}(s, \nu) = (\sin s)^{\nu+1} F\big(-\ell, \ell + \nu + 1; \tfrac{1}{2}; \cos^2 s\big),$$

where F denotes the hypergeometric function. Equivalently, using Gegenbauer polynomials $G_n^{(\lambda)}(x)$ we have [20, equation (4.7.30)]

$$W_m^{(1)}(s,\nu) = (\sin s)^{\nu+1}(-1)^\ell \binom{\ell+\nu}{\ell}^{-1} G_{2\ell}^{(\nu+1)}(\cos s).$$

The binomial coefficient may vanish but the formula remains valid if we take limits $\nu \to \nu_0$ at exceptional values $\nu = \nu_0$.

Similarly, let $w_m^{(2)}(s,\nu,k)$ be the solution of (9.1) with $h = H_m^{(2)}(\nu,k)$ satisfying the initial conditions

$$w\left(\frac{\pi}{2}\right) = 0, \qquad \frac{\mathrm{d}w}{\mathrm{d}s}\left(\frac{\pi}{2}\right) = 1.$$

We choose $\ell \in \mathbb{N}_0$ such that $H_m^{(2)}(\nu,0) = (2\ell+\nu+2)^2$. Then we obtain

$$w_m^{(2)}(s,\nu,k) \to W_m^{(2)}(s,\nu) = -(\sin s)^{\nu+1}\cos s F\left(-\ell,\ell+\nu+2;\tfrac{3}{2};\cos^2 s\right)$$

as $k \to 0^+$ uniformly on compact subintervals of $(0,\pi)$. In terms of Gegenbauer polynomials we have

$$W_m^{(2)}(s,\nu) = (\sin s)^{\nu+1}(-1)^{\ell+1}\left(2(\nu+1)\binom{\ell+\nu+1}{\ell}\right)^{-1} G_{2\ell+1}^{(\nu+1)}(\cos s).$$

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