On the Number of τ-Tilting Modules over Nakayama Algebras

Hanpeng GAO [†] *and Ralf SCHIFFLER* [‡]

[†] *Department of Mathematics, Nanjing University, Nanjing 210093, P.R. China*
   E-mail: *hpgao07@163.com*

[‡] *Department of Mathematics, University of Connecticut, Storrs, CT 06269-1009, USA*
   E-mail: *schiffler@math.uconn.edu*

Abstract. Let Λ_n^r be the path algebra of the linearly oriented quiver of type $\mathbb{A}$ with n vertices modulo the r-th power of the radical, and let $\widetilde{\Lambda}_n^r$ be the path algebra of the cyclically oriented quiver of type $\widetilde{\mathbb{A}}$ with n vertices modulo the r-th power of the radical. Adachi gave a recurrence relation for the number of τ-tilting modules over Λ_n^r. In this paper, we show that the same recurrence relation also holds for the number of τ-tilting modules over $\widetilde{\Lambda}_n^r$. As an application, we give a new proof for a result by Asai on recurrence formulae for the number of support τ-tilting modules over Λ_n^r and $\widetilde{\Lambda}_n^r$.

Key words: τ-tilting modules; support τ-tilting modules; Nakayama algebras

2020 Mathematics Subject Classification: 16G20; 16G60

1 Introduction

The starting point of *tilting theory* was the introduction of tilting modules over a hereditary algebra by Happel and Ringel in [10]. Ever since, the study of tilting modules and their endomorphism algebras has been an important branch of representation theory.

About 25 years later, *cluster-tilting theory*, introduced in [7] (and [8] in type $\mathbb{A}$), brought a new perspective to the subject by replacing the hereditary algebra by its cluster category, and a direct relation between tilted and cluster-tilted algebras was then established in [4].

In 2014, Adachi, Iyama and Reiten [2] introduced τ-*tilting theory* replacing the rigidity condition $\operatorname{Ext}_\Lambda^1(T, T) = 0$ for a tilting module by the weaker condition $\operatorname{Hom}_\Lambda(T, \tau_\Lambda T) = 0$ for a τ-tilting module, where τ denotes the Auslander–Reiten translation, and Λ is any finite-dimensional algebra. They showed that, in contrast to tilting modules but in agreement with cluster-tilting objects, it is always possible to exchange a given indecomposable summand of a support τ-tilting module for a unique other indecomposable and obtain a new support τ-tilting module. This process, called mutation, is essential in cluster theory.

In the same paper, the authors also showed that the support τ-tilting modules are in bijection with several other important classes in representation theory including functorially finite torsion classes introduced in [6], 2-term silting complexes introduced in [11], and cluster-tilting objects in the cluster category when the algebra Λ is hereditary, or, more generally, cluster-tilted.

Therefore, it is natural to ask what is the number of support τ-tilting modules over a given algebra.

For a hereditary algebra, the support τ-tilting modules are exactly the support tilting modules. Moreover they are in bijection with the cluster-tilting objects in the cluster category and

hence with the clusters in the corresponding cluster algebra. For Dynkin type, these numbers were first calculated in [9] via cluster algebras, and later in [12] via representation theory. In particular, over a hereditary algebra of type $\mathbb{A}_n$, the number of tilting modules is C_n, and the number of support tilting modules is C_{n+1} where C_i denotes the i-th Catalan number $\frac{1}{i+1}\binom{2i}{i}$.

In this paper, we study this question over finite-dimensional Nakayama algebras. Recall that a finite-dimensional K-algebra is said to be a *Nakayama algebra* if every indecomposable projective module and every indecomposable injective module has a unique composition series. Nakayama algebras come in two types, in fact, a finite-dimensional algebra is Nakayama if and only if its quiver is one of the following

$$A_n: \quad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n, \qquad \widetilde{A}_n: \quad 1 \overset{\frown}{\longrightarrow} 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n,$$

see [5, Section V.3.2]. Throughout the paper, we use the following notation

$$\Lambda_n^r = KA_n/\operatorname{rad}^r \qquad \text{and} \qquad \widetilde{\Lambda}_n^r = K\widetilde{A}_n/\operatorname{rad}^r.$$

Moreover, we let $t_r(n)$ and $\tilde{t}_r(n)$ denote the number of τ-tilting modules over Λ_n^r and $\widetilde{\Lambda}_n^r$, and let $s_r(n)$ and $\tilde{s}_r(n)$ denote the number of support τ-tilting modules over Λ_n^r and $\widetilde{\Lambda}_n^r$, respectively. We also set $t_r(n) := 0$ and $s_r(n) := 0$ for $n < 0$.

Adachi classified τ-tilting modules over Nakayama algebras in [1]. Under the assumption that the Loewy length of every indecomposable projective module is at least n, Adachi showed that the number of τ-tilting modules is exactly $\binom{2n-1}{n-1}$ and the number of support τ-tilting modules is $\binom{2n}{n}$. Moreover, he also gave the following recurrence relation for the number $t_r(n)$ of τ-tilting modules over Λ_n^r

$$t_r(n) = \sum_{i=1}^{r} C_{i-1} \cdot t_r(n-i).$$

The aim of this paper is study the number of τ-tilting modules over $\widetilde{\Lambda}_n^r$. We show that there is a close relationship between the number $\tilde{t}_r(n)$ of τ-tilting $\widetilde{\Lambda}_n^r$-modules and the number $t_r(n)$ of τ-tilting Λ_n^r-modules.

Proposition 1.1 (see Proposition 3.7)**.**

$$\tilde{t}_r(n) = \sum_{i=1}^{r} i \cdot C_{i-1} \cdot t_r(n-i).$$

Next, we prove that the functions t and $\tilde{t}$ satisfy the same recurrence relation.

Theorem 1.2 (see Theorem 3.8)**.** *We have the following recurrence relation*

$$\tilde{t}_r(n) = \sum_{i=1}^{r} C_{i-1} \cdot \tilde{t}_r(n-i).$$

As an application, we obtain a new proof for the following result by Asai on the numbers of support τ-tilting modules $s_r(n)$ and $\tilde{s}_r(n)$.

Theorem 1.3 ([3, Theorem 4.1])**.**

(1) $s_r(n) = 2s_r(n-1) + \sum\limits_{i=2}^{r} C_{i-1} \cdot s_r(n-i).$

(2) $\tilde{s}_r(n) = 2\tilde{s}_r(n-1) + \sum\limits_{i=2}^{r} C_{i-1} \cdot \tilde{s}_r(n-i).$

Asai used a bijection between support τ-tilting modules and semibricks to obtain his result in the context of semibricks. Our proof is combinatorial.

The paper is organized as follows. In Section 2, we fix the notation and recall several results of Adachi that are relevant to this paper. We study the Nakayama algebras of type A_n and $\widetilde{A}_n$ and prove Theorem 1.2 in Section 3. Theorem 1.3 is proved in Section 4. We include tables of the numbers of τ-tilting and support τ-tilting modules in Section 5.

2 Preliminaries

Throughout this paper, all algebras will be basic, connected, finite-dimensional algebras over an algebraically closed field K and all modules will be finitely generated right modules. For an algebra Λ, we denote by $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules and by τ_Λ the Auslander–Reiten translation of Λ. Let $\{e_1, e_2, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of Λ. We put $P_i = e_i\Lambda$ the indecomposable projective module and $S_i = P_i/\operatorname{rad} P_i$ the simple module of Λ for $i = 1, 2, \ldots, n$. For $M \in \operatorname{mod}\Lambda$, we denote by $\ell(M)$ the Loewy length of M and by $|M|$ the number of pairwise nonisomorphic indecomposable summands of M. For a finite set X, we denote by $|X|$ the cardinality of X. For details on representation theory of finite-dimensional algebras we refer to [5, 13].

Let Λ be an algebra. In this section, we recall results about support τ-tilting modules that are needed later.

Definition 2.1. Let $M \in \operatorname{mod}\Lambda$.

(1) M is called τ-*rigid* if $\operatorname{Hom}_\Lambda(M, \tau_\Lambda M) = 0$.

(2) M is called τ-*tilting* if it is τ-rigid and $|M| = |\Lambda|$.

(3) M is called *support τ-tilting* if it is a τ-tilting $\Lambda/\Lambda e\Lambda$-module for some idempotent e of Λ.

(4) M is called *proper support τ-tilting* if it is a support τ-tilting but not a τ-tilting Λ-module.

Recall that $M \in \operatorname{mod}\Lambda$ is called *sincere* if every simple Λ-module appears as a composition factor in M. It is well-known that the τ-tilting modules are exactly the sincere support τ-tilting modules [2, Proposition 2.2(a)].

We will denote by τ-tilt Λ (respectively, sτ-tilt Λ, psτ-tilt Λ) the set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting, proper support τ-tilting) Λ-modules. Obviously, we have $|\,$sτ-tilt $\Lambda| = |\,\tau$-tilt $\Lambda| + |\,$psτ-tilt $\Lambda|$.

Let psτ-tilt$_{np}\,\Lambda := \{M \in \text{ps}\tau\text{-tilt }\Lambda \mid M$ has no projective direct summands$\}$. We recall the following results proved by Adachi in [1].

Theorem 2.2 ([1, Theorem 2.6]). *Let Λ be a Nakayama algebra. There is a bijection between τ-tilt Λ and psτ-tilt$_{np}\,\Lambda$.*

The following result is very useful.

Proposition 2.3 ([1, Proposition 2.32]). *Let Λ be a Nakayama algebra of type A_n. Then each τ-tilting Λ-module has P_1 as a direct summand.*

The following recurrence relations are very useful to calculate the number of τ-tilting modules over Nakayama algebras of type A_n.

Lemma 2.4 ([1, Corollary 2.34]). *Let Λ be a Nakayama algebra of type A_n. Then*

$$|\,\tau\text{-tilt }\Lambda| = \sum_{i=1}^{\ell(P_1)} C_{i-1} \cdot |\,\tau\text{-tilt}(\Lambda/\langle e_{\leqslant i}\rangle)|,$$

where $e_{\leqslant i} := e_1 + e_2 + \cdots + e_i$. In particular, for $\Lambda = \Lambda_n^r$, we have

$$t_r(n) = \sum_{i=1}^r C_{i-1} \cdot t_r(n-i),$$

where $t_r(n)$ is the number of τ-tilting modules over Λ_n^r.

See Table 1 in Section 5 for explicit values of $t_r(n)$.

Remark 2.5. If $r \geq n$ then Λ_n^r is hereditary, and we have $t_r(n) = C_n$. On the other hand, Lemma 2.4 yields the equation $t_n(n) = \sum_{i=1}^n C_{i-1} \cdot t_n(n-i)$. Hence we recover the well-known combinatorial identity

$$C_n = \sum_{i=1}^n C_{i-1} \cdot C_{n-i}.$$

3 The number of $\boldsymbol{\tau}$-tilting modules over Nakayama algebras

In this section, Λ will be any Nakayama algebra of type A_n. As usual, we use the notation $t_r(n)$ for the number of τ-tiling modules over $\Lambda_n^r = KA_n/\mathrm{rad}^r$ and $\widetilde{t}_r(n)$ for the number of τ-tiling modules over $\widetilde{\Lambda}_n^r = K\widetilde{A}_n/\mathrm{rad}^r$. We will show that the functions t and $\tilde{t}$ satisfy the same recurrence relation.

We denote by W_i $(i = 1, 2, \ldots, n)$ the set of support τ-tilting Λ-modules which have the simples $S_1, S_2, \ldots, S_{i-1}$ as composition factor but not S_i. We also write $\Lambda_{>i} := \Lambda/\langle e_{\leqslant i}\rangle$ and $\Lambda_{<i} := \Lambda/\langle e_{\geqslant i}\rangle$ where $e_{\leqslant i} := e_1 + e_2 + \cdots + e_i$ and $e_{\geqslant i} := e_i + e_{i+1} + \cdots + e_n$.

Lemma 3.1. $|W_i| = |\tau\text{-tilt}\,\Lambda_{<i}| \cdot |s\tau\text{-tilt}\,\Lambda_{>i}|$.

Proof. Since the quiver of Λ is tree, we have $\Lambda/\langle e_i\rangle \cong \Lambda_{<i} \times \Lambda_{>i}$. Thus there is a bijection

$$\tau\text{-tilt}\,\Lambda_{<i} \times s\tau\text{-tilt}\,\Lambda_{>i} \longrightarrow W_i$$

given by $(N_1, N_2) \mapsto N_1 \oplus N_2$ where N_1 is a τ-tilting $\Lambda_{<i}$-module and N_2 is a support τ-tilting $\Lambda_{>i}$-module. Hence $|W_i| = |\tau\text{-tilt}\,\Lambda_{<i}| \cdot |s\tau\text{-tilt}\,\Lambda_{>i}|$. ■

Proposition 3.2. *Let Λ be a Nakayama algebra of type A_n. We have*

(1) $|\mathrm{ps}\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^n |\tau\text{-tilt}\,\Lambda_{<i}| \cdot |s\tau\text{-tilt}\,\Lambda_{>i}|,$

(2) $|s\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^n |\tau\text{-tilt}\,\Lambda_{<i}| \cdot |s\tau\text{-tilt}\,\Lambda_{>i}| + \sum_{i=1}^{\ell(P_1)} C_{i-1} \cdot |\tau\text{-tilt}\,\Lambda_{>i}|,$

(3) $|\mathrm{ps}\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^n |s\tau\text{-tilt}\,\Lambda_{<i}| \cdot |\tau\text{-tilt}\,\Lambda_{>i}|,$

(4) $|s\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^n |s\tau\text{-tilt}\,\Lambda_{<i}| \cdot |\tau\text{-tilt}\,\Lambda_{>i}| + \sum_{i=1}^{\ell(P_1)} C_{i-1} \cdot |\tau\text{-tilt}\,\Lambda_{>i}|.$

Proof. (1) Since $\mathrm{ps}\tau\text{-tilt}\,\Lambda = \bigcup_{i=1}^n W_i$, we have

$$|\mathrm{ps}\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^n |W_i| = \sum_{i=1}^n |\tau\text{-tilt}\,\Lambda_{<i}| \cdot |s\tau\text{-tilt}\,\Lambda_{>i}|$$

by Lemma 3.1.

(2) Since $|s\tau\text{-tilt}\,\Lambda| = |\mathrm{ps}\tau\text{-tilt}\,\Lambda| + |\tau\text{-tilt}\,\Lambda|$, the statement follows from Lemma 2.4.

(3) is similar to (1) and (4) follows from (3). ■

We give an example of Proposition 3.2.

Example 3.3. Let Λ be an algebra is given by the quiver $1 \overset{\alpha}{\twoheadrightarrow} 2 \overset{\beta}{\twoheadrightarrow} 3 \to 4$ with the relation $\alpha\beta = 0$.

	$\Lambda_{<i}$	$\Lambda_{>i}$	$\lvert\,\tau\text{-tilt}\,\Lambda_{<i}\rvert$	$\lvert\,s\tau\text{-tilt}\,\Lambda_{>i}\rvert$	$\lvert\,s\tau\text{-tilt}\,\Lambda_{<i}\rvert$	$\lvert\,\tau\text{-tilt}\,\Lambda_{>i}\rvert$
$i=1$	0	$2 \overset{\beta}{\twoheadrightarrow} 3 \to 4$	1	14	1	5
$i=2$	1	$3 \to 4$	1	5	2	2
$i=3$	$1 \overset{\alpha}{\twoheadrightarrow} 2$	4	2	2	5	1
$i=4$	$1 \overset{\alpha}{\twoheadrightarrow} 2 \overset{\beta}{\twoheadrightarrow} 3\,,\,\alpha\beta=0$	0	3	1	12	1

By Proposition 3.2(1), $\lvert\,\text{ps}\tau\text{-tilt}\,\Lambda\rvert = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 3 \cdot 1 = 26$. Note that $\ell(P_1) = 2$, and thus Lemma 2.4 implies $\lvert\,\tau\text{-tilt}\,\Lambda\rvert = 1 \cdot 5 + 1 \cdot 2 = 7$. Hence, we have $\lvert\,s\tau\text{-tilt}\,\Lambda\rvert = 26 + 7 = 33$ by Proposition 3.2(2). Moreover, we can also use part (3) of Proposition 3.2 and compute $\lvert\,\text{ps}\tau\text{-tilt}\,\Lambda\rvert = 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 12 \cdot 1 = 26$.

Corollary 3.4. *Let V_ℓ $(\ell = 1, 2, \ldots, n)$ be the set of all support τ-tilting Λ-modules which have $S_\ell, S_{\ell-1}, \ldots, S_1$ as composition factor. Then we have*

$$|V_\ell| = \sum_{i=\ell+1}^{n} \lvert\,\tau\text{-tilt}\,\Lambda_{<i}\rvert \cdot \lvert\,s\tau\text{-tilt}\,\Lambda_{>i}\rvert + \lvert\,\tau\text{-tilt}\,\Lambda\rvert.$$

Proof. This result follows from $V_\ell = \left(\bigcup_{i=\ell+1}^{n} W_i \right) \bigcup \tau\text{-tilt}\,\Lambda$. $\hspace{1cm}$ ∎

From now, we will study the number of τ-tilting $\widetilde{\Lambda}_n^r$-modules. The following result is very useful to calculate the number of proper support τ-tilting $\widetilde{\Lambda}_n^r$-modules.

Proposition 3.5.

$$\left\lvert\,\text{ps}\tau\text{-tilt}\,\widetilde{\Lambda}_n^r\right\rvert = \sum_{i=1}^{n-1} i \cdot t_r(i-1) \cdot s_r(n-i-1) + n \cdot t_r(n-1).$$

Proof. For $1 \leqslant \ell \leqslant n-1$, we define

$$K_{n,\ell} = \left\{ N \in s\tau\text{-tilt}\,\widetilde{\Lambda}_n^r \mid N \text{ has } S_n, S_{n-1}, \ldots, S_{n-\ell+1} \text{ as composition factor but not } S_{n-\ell} \right\}$$

and

$$K_{n,0} = \left\{ N \in s\tau\text{-tilt}\,\widetilde{\Lambda}_n^r \mid N \text{ does not contain } S_n \text{ as composition factor} \right\}.$$

Note that $\widetilde{\Lambda}_n^r / \langle e_{n-\ell} \rangle$ is the quotient of path algebra of the quiver

$$n - \ell + 1 \longrightarrow \cdots \longrightarrow n \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n - \ell - 1$$

by the 2-sided ideal generated by paths of length r. Therefore $\widetilde{\Lambda}_n^r / \langle e_{n-\ell} \rangle \cong \Lambda_{n-1}^r$. By Corollary 3.4, we have

$$|K_{n,\ell}| = |V_\ell| = \sum_{i=\ell+1}^{n-1} t_r(i-1) \cdot s_r(n-i-1) + t_r(n-1).$$

In particular, $|K_{n,0}| = s_r(n-1) = \sum_{i=1}^{n-1} t_r(i-1) \cdot s_r(n-i-1) + t_r(n-1)$. Hence,

$$\big| \,\mathrm{ps}\tau\text{-tilt } \widetilde{\Lambda}_n^r \big| = \sum_{\ell=0}^{n-1} |K_{n,\ell}| = \sum_{\ell=0}^{n-1} \Big(\sum_{i=\ell+1}^{n-1} t_r(i-1) \cdot s_r(n-i-1) + t_r(n-1) \Big)$$

$$= \sum_{i=1}^{n-1} i \cdot t_r(i-1) \cdot s_r(n-i-1) + n \cdot t_r(n-1). \qquad \blacksquare$$

We set $X_n = \varnothing$, for $n < 0$, and for all $n \geq 0$, we define

$$X_n = \big\{ N \in \mathrm{s}\tau\text{-tilt } \Lambda_n^r \,|\, N \text{ does not contain } P_{n-r+1}, P_{n-r}, \ldots, P_1 \text{ as direct summands} \big\}$$

and

$$Y_{n,\ell} = \{ N \in X_n \,|\, N \text{ contains } S_\ell, S_{\ell-1}, \ldots, S_1 \text{ as composition factor} \}, \quad \ell = 1, 2, \ldots, n.$$

In particular, $X_n = \mathrm{s}\tau\text{-tilt } \Lambda_n^r$ when $n+1 \leqslant r$, and $Y_{n,0} = X_n$.

Lemma 3.6. *With the above notions, we have*

(1) $|X_n| = t_r(n+1)$,

(2) $|Y_{n,\ell}| = \begin{cases} \sum_{i=\ell+1}^{r} C_{i-1} \cdot t_r(n-i+1) & \text{if } \ell \leqslant r-1, \\ 0 & \text{if } \ell \geqslant r. \end{cases}$

Proof. (1) Write $\Lambda = \Lambda_n^r$, we have $\Lambda/\langle e_i \rangle \cong \Lambda_{<i} \times \Lambda_{>i}$ for a given i. Let Z_1 be the set of all support τ-tilting $\Lambda_{<i}$-modules which have $S_1, S_2, \ldots, S_{i-1}$ as composition factor (they are exactly τ-tilting $\Lambda_{<i}$-modules) and they don't have $P_1, P_2, \ldots, P_{n-r+1}$ as direct summands.

If $i \leqslant r$, then $Z_1 = \tau\text{-tilt } \Lambda_{<i}$.

If $i \geqslant r+1$, then all τ-tilting $\Lambda_{<i}$-modules have P_1 as direct summand by Proposition 2.3, and hence $Z_1 = \varnothing$.

Let Z_2 be the set of all support τ-tilting $\Lambda_{>i}$-modules who do not have $P_1, P_2, \ldots, P_{n-r+1}$ as direct summands. Then Z_2 consists of exactly all support τ-tilting $\Lambda_{>i}$-modules which do not contain $P_{i+1}, P_{i+2}, \ldots, P_{n-r+1}$ as direct summands. Denoted by $X_{n,i} \subseteq X_n$ $(i = 1, 2, \ldots, n)$ the subset of all support τ-tilting modules having $S_1, S_2, \ldots, S_{i-1}$ as composition factor but not S_i and by $X_{n,n+1} \subseteq X_n$ the subset of all support τ-tilting modules having $S_1, S_2, \ldots, S_n$ as composition factor (hence, they are exactly τ-tilting). We have $|Z_2| = |X_{n-i}|$, since $\Lambda_{>i} \cong \Lambda_{n-i}^r$. There is a bijection between $Z_1 \times Z_2$ and $X_{n,i}$ given by $(N_1, N_2) \to N_1 \oplus N_2$ where $N_1 \in Z_1$ and $N_2 \in Z_2$. Therefore, $|X_{n,i}| = |Z_1| \cdot |Z_2|$. Thus, we have

$$|X_{n,i}| = \begin{cases} |\tau\text{-tilt } \Lambda_{<i}| \cdot |X_{n-i}| & \text{if } i \leqslant r, \\ 0 & \text{if } i \geqslant r+1. \end{cases}$$

If $r \leqslant n$, we have

$$|X_n| = \sum_{i=1}^{n+1} |X_{n,i}| = \sum_{i=1}^{r} |\tau\text{-tilt } \Lambda_{<i}| \cdot |X_{n-i}| = \sum_{i=1}^{r} C_{i-1} \cdot |X_{n-i}|.$$

Note that, if $r \geq n+1$ then $X_n = \mathrm{s}\tau\text{-tilt } \Lambda_n^r$ and hence $|X_n| = C_{n+1}$ since Λ_n^r is hereditary, we get $|X_n| = t_r(n+1)$. On the other hand, Lemma 2.4 says that $t_r(n)$ and $|X_n|$ satisfy the same recursive formula. Thus we have $|X_n| = t_r(n+1)$ for all n, by induction.

(2) follows from (1) and the fact $Y_{n,\ell} = \bigcup_{i=\ell+1}^{n+1} X_{n,i}$. $\qquad \blacksquare$

As a result of Lemma 3.6, we have the following relationship between $\tilde{t}_r(n)$ and $t_r(n)$.

Proposition 3.7. *We have*

$$\tilde{t}_r(n) = \sum_{i=1}^{r} i \cdot C_{i-1} \cdot t_r(n-i).$$

Proof. Considering the set $K_{n,\ell}^{np}$ consisting of all modules in $K_{n,\ell}$ which do not have projective $\widetilde{\Lambda}_n^r$-modules as direct summands.

Note that the indecomposable projective $\widetilde{\Lambda}_n^r/\langle e_{n-\ell}\rangle$-modules corresponding to the last $r-1$ points (the length of them is at most $r-1$) are not projective $\widetilde{\Lambda}_n^r$-modules, we have $\left|K_{n,\ell}^{np}\right| = |Y_{n-1,\ell}|$ since $\widetilde{\Lambda}_n^r/\langle e_{n-\ell}\rangle \cong \Lambda_{n-1}^r$. Thus,

$$
\begin{aligned}
\left| \text{ps}\tau\text{-tilt}_{np}\, \widetilde{\Lambda}_n^r \right| &= \sum_{\ell=0}^{n-1} \left|K_{n,\ell}^{np}\right| = \sum_{\ell=0}^{n-1} |Y_{n-1,\ell}| \\
&= \sum_{\ell=0}^{r-1} \sum_{i=l+1}^{r} C_{i-1} \cdot t_r(n-i) \quad \text{(by Lemma 3.6)} \\
&= \sum_{i=1}^{r} i \cdot C_{i-1} \cdot t_r(n-i).
\end{aligned}
$$

Therefore, the assertion follows from Theorem 2.2. ∎

Now, we are ready to prove our main result of this section.

Theorem 3.8. *We have*

$$\tilde{t}_r(n) = \sum_{i=1}^{r} C_{i-1} \cdot \tilde{t}_r(n-i).$$

Proof. By Proposition 3.7, we have $\tilde{t}_r(n) = \sum\limits_{\ell=1}^{r} \ell \cdot C_{\ell-1} \cdot t_r(n-\ell)$. Thus,

$$
\begin{aligned}
\tilde{t}_r(n) - \sum_{i=1}^{r} C_{i-1} \cdot \tilde{t}_r(n-i) &= \sum_{\ell=1}^{r} \ell \cdot C_{\ell-1} \cdot t_r(n-\ell) \\
&\quad - \sum_{i=1}^{r} C_{i-1} \cdot \left(\sum_{\ell=1}^{r} \ell \cdot C_{\ell-1} \cdot t_r(n-i-\ell) \right) \\
&= \sum_{\ell=1}^{r} \ell \cdot C_{\ell-1} \cdot \left(t_r(n-\ell) - \sum_{i=1}^{r} C_{i-1} \cdot t_r(n-\ell-i) \right) \\
&= 0. \quad \text{(by Lemma 2.4)}
\end{aligned}
$$

Hence, $\tilde{t}_r(n) = \sum\limits_{i=1}^{r} C_{i-1} \cdot \tilde{t}_r(n-i)$. ∎

The following proposition and its proof are similar to [3, Theorem 4.1 (3)]. For convenience, we include a proof here. We shall use the following notation.

For every positive integer r, let $F_r(X) = \sum\limits_{i=0}^{r} c_i \cdot X^{r-i}$ where $c_0 = 1$ and $c_i = -C_{i-1}$ for $i = 1, 2, \ldots, r$. Let

$$\mathbf{E}_n(X_1, X_2, \ldots, X_r) = \sum_{J \subseteq \{1,2,\ldots,r\},|J|=n} \prod_{j \in J} X_j$$

be the n-th elementary symmetric polynomial, $n = 0, 1, 2, \ldots, r$. Let

$$\mathbf{H}_n(X_1, X_2, \ldots, X_r) = \sum_{\substack{t_1, t_2, \ldots, t_r \in \mathbb{Z}_{\geqslant 0} \\ t_1 + t_2 + \cdots + t_r = n}} X_1^{t_1} X_2^{t_2} \cdots X_r^{t_r} \qquad \text{for all } n \in \mathbb{Z},$$

$$\mathbf{P}_n(X_1, X_2, \ldots, X_r) = \sum_{i=1}^{r} X_i^n \quad \text{for all } n \geqslant 1.$$

In particular, we have $\mathbf{E}_0 = 1$, $\mathbf{H}_0 = 1$, and $\mathbf{H}_n = 0$ for $n < 0$.

Proposition 3.9. *Let $\xi_1, \xi_2, \ldots, \xi_r$ be the roots (not necessarily distinct) of the polynomial $F_r(X)$. Then we have*

(1) $t_r(n) = \displaystyle\sum_{\substack{t_1, t_2, \ldots, t_r \in \mathbb{Z}_{\geqslant 0} \\ t_1 + t_2 + \cdots + t_r = n}} \xi_1^{t_1} \xi_2^{t_2} \cdots \xi_r^{t_r},$

(2) $\widetilde{t}_r(n) = \displaystyle\sum_{i=1}^{r} \xi_i^n.$

Proof. Using Vieta's formula on symmetric polynomials, we have

$$\mathbf{E}_i(\xi_1, \xi_2, \ldots, \xi_r) = (-1)^i c_i$$

for $i = 0, 1, 2, \ldots, r$. By [3, Lemma 4.8], we have

$$\sum_{i=0}^{r} c_i \mathbf{H}_{n-i}(\xi_1, \xi_2, \ldots, \xi_r) = 0.$$

On the other hand, Lemma 2.4 yields $t_r(n) = \sum_{i=1}^{r} C_{i-1} \cdot t_r(n-i)$ which implies

$$\sum_{i=0}^{r} c_i \cdot t_r(n-i) = 0.$$

Therefore

$$\sum_{i=0}^{r} c_i \cdot (t_r(n-i) - \mathbf{H}_{n-i}(\xi_1, \xi_2, \ldots, \xi_r)) = 0. \tag{3.1}$$

Note that $t_r(0) = 1 = \mathbf{H}_0(\xi_1, \xi_2, \ldots, \xi_r)$ and $t_r(n) = 0 = \mathbf{H}_n(\xi_1, \xi_2, \ldots, \xi_r)$ for $n < 0$. Therefore, using induction and equation (3.1), we see that for all n

$$t_r(n) = \mathbf{H}_n(\xi_1, \xi_2, \ldots, \xi_r) = \sum_{\substack{t_1, t_2, \ldots, t_r \in \mathbb{Z}_{\geqslant 0} \\ t_1 + t_2 + \cdots + t_r = n}} \xi_1^{t_1} \xi_2^{t_2} \cdots \xi_r^{t_r}.$$

In Proposition 3.7, we have shown the following relation

$$\tilde{t}_r(n) = \sum_{i=1}^{r} -i \cdot c_i \cdot t_r(n-i).$$

Hence,

$$\tilde{t}_r(n) = \sum_{i=1}^{r} -i \cdot c_i \cdot t_r(n-i) = \sum_{i=1}^{r} -i \cdot c_i \cdot \mathbf{H}_{n-i}(\xi_1, \xi_2, \ldots, \xi_r)$$

$$= \sum_{i=1}^{r} -i \cdot (-1)^i \cdot \mathbf{E}_i(\xi_1, \xi_2, \ldots, \xi_r) \cdot \mathbf{H}_{n-i}(\xi_1, \xi_2, \ldots, \xi_r)$$

$$= \sum_{i=1}^{r} (-1)^{i-1} \cdot i \cdot \mathbf{E}_i(\xi_1, \xi_2, \ldots, \xi_r) \cdot \mathbf{H}_{n-i}(\xi_1, \xi_2, \ldots, \xi_r)$$

$$= \mathbf{P}_n(\xi_1, \xi_2, \ldots, \xi_r) \quad \text{(by [3, Lemma 4.8])}$$

$$= \sum_{i=1}^{r} \xi_i^n. \qquad \blacksquare$$

For $r = 2$, we obtain the Fibonacci recurrence $\tilde{t}_2(n) = \tilde{t}_2(n-1) + \tilde{t}_2(n-2)$; however, with different initial conditions. Thus we obtain Lucas numbers and we have the following formula.

Corollary 3.10. $\tilde{t}_2(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$.

4 The number of support τ-tilting modules over Nakayama algebras

In this section, we will apply our results to give a new proof of a theorem by Asai, see [3, Theorem 4.1(1) and (2)]

Applying Proposition 3.2, we obtain the following recurrence relation for the number of support τ-tilting modules over Λ_n^r. See Table 2 in Section 5 for explicit values of $s_r(n)$.

Proposition 4.1 ([3, Theorem 4.1(1)]).

$$s_r(n) = 2s_r(n-1) + \sum_{i=2}^{r} C_{i-1} \cdot s_r(n-i).$$

Proof. Since $\Lambda = \Lambda_n^r$, we have $\Lambda_{<i} \cong \Lambda_{i-1}^r$ and $\Lambda_{>i} \cong \Lambda_{n-i}^r$. Thus Proposition 3.2(1) yields

$$s_r(n) = \sum_{i=1}^{n} t_r(i-1) \cdot s_r(n-i) + t_r(n).$$

Therefore,

$$s_r(n) - s_r(n-1) = \sum_{i=1}^{n} t_r(i-1) \cdot s_r(n-i) + t_r(n)$$

$$- \left(\sum_{i=1}^{n-1} t_r(i-1) \cdot s_r(n-1-i) + t_r(n-1) \right)$$

$$= s_r(n-1) + \sum_{i=1}^{n-1} (t_r(i) - t_r(i-1)) \cdot s_r(n-1-i) + (t_r(n) - t_r(n-1)).$$

By Lemma 2.4, for all i we have

$$t_r(i) = t_r(i-1) + \sum_{\ell=2}^{r} C_{\ell-1} \cdot t_r(i-\ell).$$

Using this equality in the identity above, we get

$$s_r(n) - s_r(n-1) = s_r(n-1) + \sum_{i=1}^{n-1} \sum_{\ell=2}^{r} C_{\ell-1} \cdot t_r(i-\ell) \cdot s_r(n-1-i)$$

$$+ \sum_{\ell=2}^{r} C_{\ell-1} \cdot t_r(n-\ell)$$

and since $s_r(n-\ell-i) = 0$ whenever $i > n - \ell$, we obtain

$$s_r(n) - s_r(n-1) = s_r(n-1) + \sum_{\ell=2}^{r} C_{\ell-1} \cdot \sum_{i=1}^{n-\ell} t_r(i-1) \cdot s_r(n-\ell-i)$$

$$+ \sum_{\ell=2}^{r} C_{\ell-1} \cdot t_r(n-\ell)$$

$$= s_r(n-1) + \sum_{\ell=2}^{r} C_{\ell-1} \cdot \left(\sum_{i=1}^{n-\ell} t_r(i-1) \cdot s_r(n-\ell-i) + t_r(n-\ell) \right)$$

$$= s_r(n-1) + \sum_{\ell=2}^{r} C_{\ell-1} \cdot s_r(n-\ell) \quad \text{by Proposition 3.2(1).}$$

Hence, $s_r(n) = 2s_r(n-1) + \sum_{i=2}^{r} C_{i-1} \cdot s_r(n-i)$.   ∎

Remark 4.2. If $r \geq n$ then Λ_n^r is a hereditary algebra of Dynkin type $\mathbb{A}_n$ and the support τ-tilting modules are in bijection with the clusters in the corresponding cluster algebra. In particular, $s_r(n) = C_{n+1}$. This fact also can be obtained directly from Proposition 4.1 and the identity $C_n = \sum_{i=1}^{n} C_{i-1} \cdot C_{n-i}$ (see Remark 2.5). Indeed, the following equation

$$s_n(n) - C_{n+1} = 2s_n(n-1) + \sum_{i=2}^{n} C_{i-1} \cdot s_n(n-i) - \left(\sum_{i=1}^{n+1} C_{i-1} \cdot C_{n+1-i} \right)$$

$$= C_0 \cdot s_n(n-1) + \sum_{i=2}^{n} C_{i-1} \cdot s_n(n-i) - \left(\sum_{i=1}^{n} C_{i-1} \cdot C_{n+1-i} \right)$$

$$+ s_n(n-1) - C_n \cdot C_0$$

$$= \sum_{i=1}^{n} C_{i-1} \cdot (s_n(n-i) - C_{n+1-i}) + (s_n(n-1) - C_n \cdot C_0)$$

$$= \sum_{i=1}^{n} C_{i-1} \cdot (s_{n-i}(n-i) - C_{n+1-i}) + (s_{n-1}(n-1) - C_n)$$

implies $s_n(n) = C_{n+1}$ by induction.

Applying Theorem 3.8, we obtain the following recurrence relation for the number of support τ-tilting modules over $\widetilde{\Lambda}_n^r$. See Table 4 in Section 5 for explicit values of $\widetilde{s}_r(n)$.

Proposition 4.3 ([3, Theorem 4.1(2)])**.**

$$\tilde{s}_r(n) = 2\tilde{s}_r(n-1) + \sum_{i=2}^{r} C_{i-1} \cdot \tilde{s}_r(n-i).$$

Proof. We need to show

$$\tilde{s}_r(n) - 2\tilde{s}_r(n-1) - \sum_{i=2}^{r} C_{i-1} \cdot \tilde{s}_r(n-i) = 0.$$

Using Proposition 3.5 and the equation $\tilde{s}_r(n) = \big| \mathrm{ps}\tau\text{-tilt}\,\widetilde{\Lambda}_n^r \big| + \tilde{t}_r(n)$, we obtain

$$\tilde{s}_r(n) = \sum_{\ell=1}^{n-1} \ell \cdot t_r(\ell-1) \cdot s_r(n-\ell-1) + n \cdot t_r(n-1) + \tilde{t}_r(n).$$

Therefore,

$$\tilde{s}_r(n) - 2\tilde{s}_r(n-1) - \sum_{i=2}^{r} C_{i-1} \cdot \tilde{s}_r(n-i)$$

$$= \sum_{\ell=1}^{n-1} \ell \cdot t_r(\ell-1) \cdot s_r(n-\ell-1) + n \cdot t_r(n-1) + \tilde{t}_r(n)$$

$$- 2 \left(\sum_{\ell=1}^{n-2} \ell \cdot t_r(\ell-1) \cdot s_r(n-\ell-2) + (n-1) \cdot t_r(n-2) + \tilde{t}_r(n-1) \right)$$

$$- \sum_{i=2}^{r} C_{i-1} \cdot \left(\sum_{\ell=1}^{n-i-1} \ell \cdot t_r(\ell-1) \cdot s_r(n-i-\ell-1) + (n-i) \cdot t_r(n-i-1) + \tilde{t}_r(n-i) \right).$$

Note that every term in the summation $\sum\limits_{i=2}^{r} C_{i-1} \sum\limits_{\ell=n-i}^{n-2} \ell \cdot t_r(\ell-1) \cdot s_r(n-i-\ell-1)$ is zero, and therefore the whole expression is equal to

$$= \sum_{\ell=1}^{n-2} \ell \cdot t_r(\ell-1) \cdot \left(s_r(n-\ell-1) - 2s_r(n-\ell-2) - \sum_{i=2}^{r} C_{r-1} \cdot s_r(n-\ell-1-i) \right)$$

$$+ n \cdot t_r(n-1) + \left((n-1) \cdot t_r(n-2) - 2(n-1) \cdot t_r(n-2) \right)$$

$$- \sum_{i=2}^{r} (n-i) \cdot C_{i-1} \cdot t_r(n-i-1) + \left(\tilde{t}_r(n) - 2\tilde{t}_r(n-1) - \sum_{i=2}^{r} C_{i-1} \cdot \tilde{t}_r(n-i) \right).$$

Now, the parenthesis in the first sum is zero, by Proposition 4.1, the large parenthesis in the second row can be included in the first sum of the third row as the $i=1$ term, and the parenthesis in the third row is equal to $-\tilde{t}_r(n-1)$ by Theorem 3.8. So the whole expression is equal to

$$= n \cdot t_r(n-1) - \sum_{i=1}^{r} (n-i) \cdot C_{i-1} \cdot t_r(n-i-1) - \tilde{t}_r(n-1)$$

$$= n \cdot \left(t_r(n-1) - \sum_{i=1}^{r} C_{i-1} \cdot t_r(n-i-1) \right) + \sum_{i=1}^{r} i \cdot C_{i-1} \cdot t_r(n-i-1) - \tilde{t}_r(n-1)$$

$$= \sum_{i=1}^{r} i \cdot C_{i-1} \cdot t_r(n-i-1) - \tilde{t}_r(n-1) \quad \text{(by Lemma 2.4)}$$

$$= \tilde{t}_r(n-1) - \tilde{t}_r(n-1) \quad \text{(by Proposition 3.7)}$$

$$= 0. \qquad \blacksquare$$

5   Examples

In this section, we give examples of the numbers of (support) τ-tilting modules over Λ_n^r and $\widetilde{\Lambda}_n^r$ (see Tables 1–4).

Table 1. The number of τ-tilting modules of Λ_n^r.

$t_r(n) \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
r												
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	5	8	13	21	34	55	89	144	233
3	1	2	5	9	18	37	73	146	293	585	1170	2341
4	1	2	5	14	28	62	143	331	738	1665	3780	8576
5	1	2	5	14	42	90	213	527	1326	3317	8022	19608
6	1	2	5	14	42	132	297	737	1914	5081	13566	35862

Table 2. The number of support τ-tilting modules of Λ_n^r.

$s_r(n) \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
r												
1	2	4	8	16	32	64	128	256	512	1024	2048	4096
2	2	5	12	29	70	169	408	985	2378	5741	13860	33461
3	2	5	14	37	98	261	694	1845	4906	13045	34686	92229
4	2	5	14	42	118	331	934	2645	7476	21120	59676	168649
5	2	5	14	42	132	387	1130	3317	9786	28932	85352	251613
6	2	5	14	42	132	429	1298	3905	11802	35862	109376	333933

Table 3. The number of τ-tilting modules of $\widetilde{\Lambda}_n^r$.

$\tilde{t}_r(n) \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
r												
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	4	7	11	18	29	47	76	123	199	322
3	1	3	10	15	31	66	127	255	514	1023	2047	4098
4	1	3	10	35	56	126	302	715	1549	3498	7897	18158
5	1	3	10	35	126	210	498	1275	3313	8398	19691	48062
6	1	3	10	35	126	462	792	1947	5203	14278	39095	104006

Table 4. The number of support τ-tilting modules of $\widetilde{\Lambda}_n^r$.

$\tilde{s}_r(n) \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
r												
1	2	4	8	16	32	64	128	256	512	1024	2048	4096
2	2	6	14	34	82	198	478	1154	2786	6726	16238	39202
3	2	6	20	50	132	354	940	2498	6644	17666	46972	124898
4	2	6	20	70	182	504	1430	4078	11504	32466	91742	259348
5	2	6	20	70	252	672	1920	5646	16796	49966	147028	432724
6	2	6	20	70	252	924	2508	7326	22088	67606	208012	638356

Acknowledgements

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