Integral Structure for Simple Singularities

Todor MILANOV and Chenghan ZHA

Kavli IPMU (WPI), UTIAS, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan
E-mail: *todor.milanov@ipmu.jp, chenghan.zha@ipmu.jp*

Abstract. We compute the image of the Milnor lattice of an ADE singularity under a period map. We also prove that the Milnor lattice can be identified with an appropriate relative K-group defined through the Berglund–Hübsch dual of the corresponding singularity.

Key words: simple singularities; period map; mirror symmetry; topological K-theory

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1 Introduction

Let $f \in \mathbb{C}[x_1, x_2, x_3]$ be a weighted homogeneous polynomial representing the germ of a simple singularity of type A, D, or E. Let $f^T \in \mathbb{C}[x_1, x_2, x_3]$ be the corresponding Berglund–Hübsch dual of f (see Section 1.2). Fan–Jarvis–Ruan proved in [6], using also results of Givental–Milanov [10] and Frenkel–Givental–Milanov [7], that the generating function of Fan–Jarvis–Ruan–Witten (FJRW) invariants of f^T can be identified with a tau-function of a specific Kac–Wakimoto hierarchy. The identification however involves rescaling the dynamical variables of the Kac–Wakimoto hierarchy and the precise values of the rescaling constants were left unknown. One application of the results in this papers is to obtain explicit formulas for the rescaling constants. Such an explicit identification is needed if one is interested in constructing a matrix model for the FJRW invariants of f^T, similar to the Kontsevich's matrix model in [13]. We are not going to compute the rescaling coefficients in this paper. The computation is straightforward and it should probably be done only when needed. Let us try to explain instead why this small technical detail leads to a very interesting problem in singularity theory.

Let us recall that for any singularity f there is a natural way to construct a semi-simple Frobenius structure on the space of miniversal deformations of f (see [11]). The construction depends on the choice of a *primitive form* in the sense of Saito [16] and it essentially coincides with what Saito called *flat structure*. On the other hand, motivated by Gromov–Witten theory, Givental introduced the notion of a *total descendent potential* for every semi-simple Frobenius manifold (see [8, 9]). Givental conjectured [9] and Teleman proved [19] that if the Frobenius structure corresponding to the quantum cohomology of a compact Kähler manifold X is semi-simple, then his definition coincides with the generating function of Gromov–Witten invariants of X. Let us return to our settings, i.e., the case of a simple weighted homogeneous singularity f on 3 variables. The standard holomorphic volume form $dx_1 \wedge dx_2 \wedge dx_3$ is primitive. Therefore, following Givental, we can define total descendent potential. The latter will be called, the total descendent potential of f. Fan–Jarvis–Ruan proved in [6] that the generating function of FJRW invariants of f^T coincides with the *total descendant potential* of f. Furthermore, Givental–Milanov [10] and Frenkel–Givental–Milanov [7] proved that the total descendant potential of f is a tau-function of the principal Kac–Wakimoto hierarchy of the same type A, D, or E as the singularity f. Finally, the outcome of the above work is that the generating function of FJRW

invariants of f^T is a tau-function of an appropriate Kac–Wakimoto hierarchy. However, there is still a small gap in this statement. Namely, while the state space of FJRW theory is identified explicitly with the Milnor ring of the singularity (see [6]), the identification of the Milnor ring and the Cartan subalgebra of the corresponding simple Lie algebra is given by a period map and it is not explicit. In order to obtain an explicit identification, we need to determine the image of the root lattice in the Milnor ring of the singularity. This is exactly the problem that we want to solve in this paper.

1.1   Simple singularities

Let us give a precise statement of the problem that we want to solve. Let $f(x_1, x_2, x_3) = g(x_1, x_2) + x_3^2$, where g is one of the polynomials listed in the following table:

Type	A_N	D_N	E_6	E_7	E_8
g	$x_1^{N+1} + x_2^2$	$x_1^2 x_2 + x_2^{N-1}$	$x_1^3 + x_2^4$	$x_1^3 + x_1 x_2^3$	$x_1^3 + x_2^5$

The polynomial f represents the germ of a simple singularity at $x = 0$. Let

$$H_f := \mathbb{C}[x_1, x_2, x_3]/(f_{x_1}, f_{x_2}, f_{x_3})$$

be the *Milnor ring* of f, where $f_{x_i} := \frac{\partial f}{\partial x_i}$. Let us denote by $(\ ,\ )$ the *residue pairing* on H_f corresponding to the standard volume form $\omega = dx_1 \wedge dx_2 \wedge dx_3$, that is,

$$(\phi_1(x), \phi_2(x)) := \mathrm{Res}_{x=0} \frac{\phi_1(x)\phi_2(x)\omega}{f_{x_1} f_{x_2} f_{x_3}}.$$

The hypersurfaces $V_\lambda = \{x \in \mathbb{C}^3 \,|\, f(x) = \lambda\}$ for $\lambda \neq 0$ are non-singular and their union has a structure of a smooth fibration on $\mathbb{C} \setminus \{0\}$ known as the *Milnor fibration*. Let us fix a reference point $\lambda = 1$ and consider the middle homology group $H_2(V_1; \mathbb{Z})$, known also as the *Milnor lattice*. Our interest is in the period vectors $I_\alpha^{(-1)}(\lambda) \in H_f$ defined by

$$\left(I_\alpha^{(-1)}(\lambda), \phi_i\right) := \frac{1}{2\pi} \int_{\alpha_\lambda} \phi_i(x) \frac{\omega}{df},$$

where $\alpha \in H_2(V_1; \mathbb{C})$, $\phi_i(x)$ $(1 \leq i \leq N)$ is a set of polynomials representing a basis of H_f, $\alpha_\lambda \in H_2(V_\lambda; \mathbb{C})$ is obtained from α via a parallel transport along some reference path, and $\frac{\omega}{df}$ is the so-called *Gelfand–Leray* form (see [1]). Alternatively, we can view each period vector as a multivalued analytic function $I_\alpha^{(-1)} \colon \mathbb{C} \setminus \{0\} \to H_f$.

Let us assign degree $c_i \in \mathbb{Q}_{>0}$ to x_i $(1 \leq i \leq 3)$, such that, the polynomial f has degree 1. Then the Milnor ring becomes a graded ring. The highest possible degree of a homogeneous element in H_f is $D = \sum_{i=1}^{3}(1 - 2c_i) = 1 - \frac{2}{h}$, where h is the Coxeter number of the corresponding root system. Put $\theta := \frac{D}{2} - \deg$, where $\deg \colon H_f \to H_f$ is the linear operator uniquely determined by the following condition: if ϕ is a weighted homogeneous element of degree d, then $\deg(\phi) = d\phi$. For homogeneity reasons, the period vectors have the form

$$I_\alpha^{(-1)}(\lambda) = \frac{\lambda^{\theta + 1/2}}{\Gamma(\theta + 3/2)} \Psi(\alpha), \tag{1.1}$$

where $\Psi \colon H_2(V_1; \mathbb{C}) \to H_f$ is a linear isomorphism. Our goal is to compute the image of the Milnor lattice $H_2(V_1; \mathbb{Z})$ via the map Ψ. The solution to this problem is given in Section 2. Explicit formulas for the image of the Milnor lattice via the map Ψ are given in Sections 2.3–2.7. The main feature of our answer is that it involves various Γ-constants and roots of unity. The second goal of our paper is to show that although the formulas look cumbersome, in fact there is an interesting structure behind them.

1.2 K-theoretic interpretation of the Milnor lattice

It turns out that our answer can be stated quite elegantly via relative K-theory. The idea to look for such a description comes from the work of Iritani [12], Chiodo–Iritani–Ruan [4], and Chiodo–Nagel [5]. More precisely, Iritani was able to prove in [12] that the Milnor lattice of the mirror of a Fano toric orbifold X can be identified with the topological K-ring $K^0(X)$. The identification uses a period map to embed the Milnor lattice in $H^*(X;\mathbb{C})$ and a certain Γ-class modification of the Chern character map to embed $K^0(X)$ in $H^*(X;\mathbb{C})$. The lattice in $H^*(X;\mathbb{C})$, obtained either as the image of the Milnor lattice via the period map or as the image of $K^0(X)$ via the Γ-class modification of the Chern character map, is known as Γ-integral structure in quantum cohomology. Isolated singularities are almost never mirror models of a manifold. Nevertheless, Chiodo–Iritani–Ruan have proposed an analogue of the Γ-integral structure for singularities of Fermat type. The analogue of $H^*(X;\mathbb{C})$ is played by the Milnor ring H_f, while $K^0(X)$ is replaced with an appropriate category of equivariant matrix factorizations of f. Finally, Chiodo–Nagel were able to find an isomorphism between H_f and an appropriate relative orbifold cohomology group. Since, the Chern character gives an isomorphism between cohomology and K-theory and the Grothendieck group of the category of matrix factorization also has the flavor of a topological K-ring, after expecting more carefully the constructions in [4] and [5], we see that there is a natural candidate for a Γ-integral structure for Fermat type singularities. After several trial and errors we were able to find the correct topological K-ring and the correct modification of the Chern character map. Moreover our proposal makes sense not only for Fermat type polynomials, but more generally for an arbitrary *invertible polynomial*. Nevertheless, let us return to our current settings of simple singularities. We believe that our results can be generalized to all invertible polynomials, but that would require some additional work.

The polynomials f corresponding to a simple singularity are invertible polynomials in the sense of [3] (see also [14]). Each polynomial is uniquely determined by a 3×3 matrix $A = (a_{ij})_{1 \leq i,j \leq 3}$ with non-negative integer coefficients, such that,

$$f(x) = \sum_{i=1}^{3} x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}}.$$

Following Fan–Jarvis–Ruan (see [6]) we consider also the Berglund–Hübsch dual polynomial

$$f^T(x) = \sum_{i=1}^{3} x_1^{a_{1i}} x_2^{a_{2i}} x_3^{a_{3i}}.$$

Let G^T be the group of diagonal symmetries of f^T, that is,

$$G^T := \big\{ t \in (\mathbb{C}^*)^3 \,|\, t_1^{a_{1i}} t_2^{a_{2i}} t_3^{a_{3i}} = 1 \;\forall\, i \big\}.$$

Let a^{ij} ($1 \leq i,j \leq 3$) be the entries of the inverse matrix A^{-1}. The group G^T is generated by the following elements

$$\overline{\rho}_i = \big(e^{2\pi \mathbf{i} a^{i1}}, e^{2\pi \mathbf{i} a^{i2}}, e^{2\pi \mathbf{i} a^{i3}} \big), \qquad 1 \leq i \leq 3.$$

Finally, let $V_1^T = \big\{ x \in \mathbb{C}^3 \,|\, f^T(x) = 1 \big\}$. Our main interest is in the topological relative K-theoretic orbifold group

$$K^0_{\mathrm{orb}}\big(\big[\mathbb{C}^3 / G^T\big], \big[V_1^T / G^T \big] \big) := K^0_{G^T}\big(\mathbb{C}^3, V_1^T \big).$$

In general, there is no satisfactory definition of K-theory for non-compact spaces. However, in our case the pair $\big(\mathbb{C}^3, V_1^T \big)$ is G^T-equivariantly homotopic to a pair of finite CW complexes, so

we may think of $(\mathbb{C}^3, V_1^T)$ as a G^T-equivariant pair of finite CW-complexes. We refer to [18] for some background on equivariant topological K-theory.

Motivated by Iritani's Γ-integral structure in quantum cohomology (see [12]), we will now construct a linear map

$$\mathrm{ch}_\Gamma \colon \quad K_{\mathrm{orb}}^0\big([\mathbb{C}^3/G^T], [V_1^T/G^T]\big) \otimes \mathbb{C} \longrightarrow H_{\mathrm{orb}}\big([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{C}\big), \qquad (1.2)$$

which is a certain Γ-class modification of the orbifold Chern character map. For a G^T-equivariant space X and $g \in G^T$, let us denote by $\mathrm{Fix}_g(X) := \{x \in X \mid gx = x\}$ the set of fixed points. The elements in the relative K-group will be identified with isomorphism classes $[E \to F]$ of two-term complexes $E \xrightarrow{\ d\ } F$ of G^T-equivariant vector bundles, such that, the differential d is a morphism of G^T-equivariant vector bundles and $d|_{V_1^T} \colon E_{V_1^T} \to F_{V_1^T}$ is an isomorphism. Note that for $g \in G^T$, the restriction of a vector bundle $E|_{\mathrm{Fix}_g(\mathbb{C}^3)}$ decomposes as a direct sum of eigen-subbundles E_ζ and that the restriction to $\mathrm{Fix}_g(\mathbb{C}^3)$ of every two term complex $E \xrightarrow{\ d\ } F$ decomposes as a direct sum of two term subcomplexes $E_\zeta \xrightarrow{\ d_\zeta\ } F_\zeta$, where $d_\zeta = d|_{E_\zeta}$. We have the following well known decomposition (e.g., see [2, Theorem 2]):

$$\mathrm{Tr} \colon \quad K_{G^T}^0\big(\mathbb{C}^3, V_1^T\big) \otimes \mathbb{C} \xrightarrow{\ \cong\ } \bigoplus_{g \in G^T} \big[K^0\big(\mathrm{Fix}_g\big(\mathbb{C}^3\big), \mathrm{Fix}_g\big(V_1^T\big)\big) \otimes \mathbb{C}\big]^{G^T},$$

where $[\ ]^{G^T}$ denotes the G^T-invariant part and the morphism Tr is defined by

$$\mathrm{Tr}([E \to F]) = \bigoplus_{g \in G^T} \bigoplus_{\zeta \in \mathbb{C}^*} \zeta[E_\zeta \to F_\zeta].$$

Remark 1.1. The above decomposition is proved in [2] in the case of absolute K-theory. However, using the long exact sequence of a pair, it is straightforward to extend the result to relative K-theory as well.

The standard Chern character map gives an isomorphism

$$\mathrm{ch} \colon \quad K^0\big(\mathrm{Fix}_g\big(\mathbb{C}^3\big), \mathrm{Fix}_g\big(V_1^T\big)\big) \otimes \mathbb{C} \longrightarrow H^{\mathrm{ev}}\big(\mathrm{Fix}_g\big(\mathbb{C}^3\big), \mathrm{Fix}_g\big(V_1^T\big); \mathbb{C}\big).$$

Finally, if G is a finite group acting on a smooth manifold M, such that the quotient groupoid $[M/G]$ is an effective orbifold, then $H^*(M/G; \mathbb{C}) \cong [H^*(M; \mathbb{C})]^G$. Indeed, for a finite group G the operation taking G-invariants is an exact functor from the category of G-vector spaces to the category of vector spaces. Therefore

$$H^i(M/G; \mathbb{C}) \cong H^i\big([\Gamma(M, \mathcal{A}_M^*)]^G\big) = [H^i(M, \mathcal{A}_M^*)]^G \cong [H^i(M; \mathbb{C})]^G,$$

where $\mathcal{A}_M^*$ is the sheaf of smooth differential forms on M with complex coefficients, the first isomorphism is Satake's de Rham theorem for orbifolds (see [17]), and the last one is the de Rham's theorem for the manifold M. Using the long exact sequence of a pair, we get also that $H^i(M/G, N/G; \mathbb{C}) \cong [H^i(M, N; \mathbb{C})]^G$ for any G-invariant submanifold $N \subset M$. On the other hand, by definition,

$$H_{\mathrm{orb}}^*\big([\mathbb{C}^3/G^T], [V_1^T/G^T]; k\big) = \bigoplus_{g \in G^T} H^*\big(\mathrm{Fix}_g\big(\mathbb{C}^3\big)/G^T, \mathrm{Fix}_g\big(V_1^T\big)/G^T; k\big), \qquad k = \mathbb{Q}, \mathbb{R}, \mathbb{C}.$$

Therefore, the composition $\widetilde{\mathrm{ch}} := \mathrm{ch} \circ \mathrm{Tr}$ defines a ring homomorphism

$$\widetilde{\mathrm{ch}} \colon \quad K_{\mathrm{orb}}^0\big([\mathbb{C}^3/G^T], [V_1^T/G^T]\big) \otimes \mathbb{C} \longrightarrow H_{\mathrm{orb}}^{\mathrm{ev}}\big([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{C}\big),$$

which is the orbifold version of the Chern character map. Clearly $\widetilde{\mathrm{ch}}$ is an isomorphism over $\mathbb{C}$.

Remark 1.2. Orbifold cohomology H^*_{orb} has two natural gradings – standard topological degree grading coming from the topological space underlying the orbit space and Chen–Ruan grading. In this paper we work with the topological grading and the topological cup product.

Let us recall also the definition of the Γ-class. If $E \in K^0_{\mathrm{orb}}\big([\mathbb{C}^3/G^T]\big) := K^0_{G^T}(\mathbb{C}^3)$ is an orbifold vector bundle and $\mathrm{Tr}(E) = \sum_g \sum_\zeta \zeta E_\zeta$, then each eigenvalue $\zeta = e^{2\pi \mathbf{i}\alpha}$, where $0 \le \alpha < 1$ is a rational number and we define

$$\widehat{\Gamma}(E) = \sum_g \prod_{\zeta = e^{2\pi \mathbf{i}\alpha}} \prod_{i=1}^{\mathrm{rk}(E_\zeta)} \Gamma(1 - \alpha + \delta_{\zeta,i}) \in H^{\mathrm{ev}}_{\mathrm{orb}}\big([\mathbb{C}^3/G^T]\big),$$

where $\delta_{\zeta,i}$ ($1 \le i \le \mathrm{rk}(E_\zeta)$) are the Chern roots of the vector bundle E_ζ. If $E = \big[T\mathbb{C}^3/G^T\big]$ is the orbifold tangent bundle, then the Γ-class is denoted by $\widehat{\Gamma}\big([\mathbb{C}^3/G^T]\big)$. The map (1.2) is defined by the following formula:

$$\mathrm{ch}_\Gamma([E \to F]) := \frac{1}{2\pi}\widehat{\Gamma}\big([\mathbb{C}^3/G^T]\big) \cup (2\pi \mathbf{i})^{\deg_\mathbb{C}} \iota^* \widetilde{\mathrm{ch}}([E \to F]),$$

where $\deg_\mathbb{C}(\phi) = i\phi$ for $\phi \in H^{2i}_{\mathrm{orb}}\big([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{C}\big)$ and ι^* is an involution in orbifold cohomology that exchanges the direct summands corresponding to g and g^{-1}. Note that the definition of ι^* makes sense because $\mathrm{Fix}_g = \mathrm{Fix}_{g^{-1}}$.

Theorem 1.3. *There exists a linear isomorphism*

$$\mathrm{mir}\colon \quad H_f \longrightarrow H^*_{\mathrm{orb}}\big([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{C}\big),$$

such that, the map

$$\mathrm{mir}^{-1} \circ \mathrm{ch}_\Gamma\colon \quad K^0_{\mathrm{orb}}\big([\mathbb{C}^3/G^T], [V_1^T/G^T]\big) \overset{\cong}{\longrightarrow} \Psi\big(H_2\big(f^{-1}(1); \mathbb{Z}\big)\big)$$

is an isomorphism of Abelian groups.

Unfortunately we do not have a conceptual definition of the map mir. Our definition is on a case by case basis. We expect that $H^*_{\mathrm{orb}}\big([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{C}\big)$ has a natural identification with the state space of FJRW-theory under which mir is identified with the mirror map of Fan–Jarvis–Ruan (see [6]). Let us point out also that in all cases the following two properties are satisfied:

1. If $x_1^{m_1} x_2^{m_2} x_3^{m_3}$ is a homogeneous monomial representing a vector in H_f, then its image under mir is in the twisted sector corresponding to $g = \bar\rho_1^{m_1+1} \bar\rho_2^{m_2+1} \bar\rho_3^{m_3+1}$.

2. The map mir is defined over $\mathbb{Q}$, that is, mir provides an isomorphism

$$\mathbb{Q}[x_1, x_2, x_3]/(f_{x_1}, f_{x_2}, f_{x_3}) \cong H^*_{\mathrm{orb}}\big([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{Q}\big).$$

2   Period map image of the Milnor lattice

2.1   Suspension isomorphism in vanishing homology

We will reduce the problem of computing periods of the hypersurface V_λ to computing periods of the Riemann surfaces

$$M_\mu := \big\{(x_1, x_2) \in \mathbb{C}^2 \,|\, g(x_1, x_2) = \mu\big\}.$$

Consider the map $V_\lambda \to \mathbb{C}$, $(x_1, x_2, x_3) \mapsto g(x_1, x_2)$. The fibers of this map are given by

$$V_{\lambda,\mu} := M_\mu \times \left\{ -\sqrt{\lambda - \mu}, \sqrt{\lambda - \mu} \right\}.$$

Suppose now that $A \in H_1(M_\lambda; \mathbb{Z})$ is any cycle. The following two maps

$$\phi_\pm \colon \ A \times [0,1] \to V_\lambda, \qquad (x_1, x_2, t) \mapsto \left(t^{c_1} x_1, t^{c_2} x_2, \pm \sqrt{\lambda(1-t)} \right)$$

have images that fit together and give a two-dimensional cycle $\alpha \in V_\lambda$, that is, $\alpha = \Sigma A$ is the suspension of the cycle A. It is known that the above suspension operation $\Sigma \colon H_1(M_\lambda; \mathbb{Z}) \to H_2(V_\lambda; \mathbb{Z})$ is an isomorphism (see [1, Theorem 2.9]).

Note that we may choose the basis of H_f to be such that $\phi_i = \phi_i(x_1, x_2)$ does not depend on x_3. Then the integral

$$\frac{1}{2\pi} \int_{\alpha_\lambda} \phi_i \frac{\omega}{df} = \frac{1}{2\pi} \partial_\lambda \int_{\alpha_\lambda} d^{-1}(\phi_i \omega) = \frac{1}{2\pi} \partial_\lambda \int_{\alpha_\lambda} x_3 \phi_i(x_1, x_2) dx_1 \wedge dx_2,$$

where in the first equality we used the Stoke's theorem (see [1, Lemma 7.2]). Using Fubini's theorem (see [1, Lemma 7.2]), we have

$$\int_{\alpha_\lambda} x_3 \phi_i(x_1, x_2) dx_1 \wedge dx_2 = \int_0^\lambda (\lambda - \mu)^{1/2} \int_{A_\mu} \frac{\phi_i(x_1, x_2) dx_1 dx_2}{dg} d\mu$$
$$- \int_\lambda^0 (-(\lambda - \mu)^{1/2}) \int_{A_\mu} \frac{\phi_i(x_1, x_2) dx_1 dx_2}{dg} d\mu,$$

where the first integral represents integrating over $\phi_+(A \times [0,1])$, the second one over $\phi_-(A \times [0,1])$, and $A_\mu \in H_1(M_\mu)$ for $\mu = \lambda t$ is obtained from A via the rescaling $(x_1, x_2) \mapsto \left(t^{c_1} x_1, t^{c_2} x_2 \right)$. We get

$$\frac{1}{2\pi} \int_{\alpha_\lambda} \phi_i \frac{\omega}{df} = \frac{1}{\pi} \partial_\lambda \int_0^\lambda (\lambda - \mu)^{1/2} \int_{A_\mu} \frac{\phi_i(x_1, x_2) dx_1 dx_2}{dg} d\mu. \tag{2.1}$$

The image of the Milnor lattice $H_2(V_1; \mathbb{Z})$ will be computed with formula (2.1).

2.2 Simple singularities and root systems

Let us first recall several well known facts about simple singularities, which will be needed in our computation (see [1, Theorem 3.14]). The analytic continuation of $I_\alpha^{(-1)}(\lambda)$ along a loop around $\lambda = 0$ yields $I_{\sigma(\alpha)}^{(-1)}(\lambda)$, where $\sigma \colon H_2(V_1; \mathbb{Z}) \to H_2(V_1; \mathbb{Z})$ is the so-called *classical monodromy* operator. Recalling the definition of Ψ (see formula (1.1)), we get the following relation:

$$\Psi(\sigma(\alpha)) = -e^{2\pi \mathbf{i}\theta} \Psi(\alpha), \tag{2.2}$$

where $\mathbf{i} := \sqrt{-1}$. In particular, knowing the image of one cycle α allows us to find the image of the entire σ-orbit of α.

Let us define

$$(\alpha|\beta) := \lambda \big(I_\alpha^{(0)}(\lambda), I_\beta^{(0)}(\lambda) \big),$$

where $I_\alpha^{(0)}(\lambda) := \partial_\lambda I_\alpha^{(-1)}(\lambda)$. It is straightforward to check that

$$(\alpha|\beta) = \frac{1}{\pi} (\Psi(\alpha), \cos(\pi\theta) \Psi(\beta)). \tag{2.3}$$

It is known that $(\alpha|\beta) = -\alpha \circ \beta$, where $\circ$ is the intersection pairing (see [10, 16]). In particular, the form $(\ |\ )$ takes integer values on the Milnor lattice.

Finally, let us also recall that we have the following remarkable facts (see [1, Theorem 3.14]):

1. The set of vanishing cycles of the singularity f coincides with the set of all $\alpha \in H_2(V_1; \mathbb{Z})$ such that $(\alpha|\alpha) = 2$.

2. The triple (Milnor lattice, set of vanishing cycles, pairing $(\ |\ )$) form a root system of the same type as the type of the singularity f, that is, the set of vanishing cycles corresponds to the roots, the Milnor lattice corresponds to the root lattice, and $(\ |\ )$ corresponds to the invariant bilinear form.

3. The classical monodromy corresponds to a Coxeter transformation.

2.3　A_N-singularity

Let us fix the following basis of H_f:

$$\phi_i = x_1^{i-1}, \qquad 1 \leqslant i \leqslant N.$$

The residue pairing takes the form

$$(\phi_i, \phi_j) = \frac{1}{4h}\delta_{i+j,h}, \qquad 1 \leqslant i,j \leqslant N,$$

where $h = N + 1$ is the Coxeter number. The Riemann surface M_μ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by the equation $x_1^{N+1} + x_2^2 = \mu$. The projection $(x_1, x_2) \mapsto x_1$ defines a degree 2 branched covering $M_\mu \to \mathbb{C}$, with branching points $x_{1,k} = \mu^{\frac{1}{N+1}}\eta_{N+1}^k$ $(k \in \mathbb{Z}_{N+1})$, where $\eta_{N+1} := e^{2\pi\mathbf{i}/(N+1)}$ and $\mathbb{Z}_{N+1} := \mathbb{Z}/(N+1)\mathbb{Z}$.

Let us construct a basis of $H_1(M_\mu; \mathbb{Z}) \cong \mathbb{Z}^N$. Cycles on M_μ can be visualized easily via their projections on the x_1-plane $\mathbb{C}$. Let L_k $(k \in \mathbb{Z}_{N+1})$ be the line segment $[0, x_{1,k}]$ (in the x_1-plane). Let A'_k be a loop in the x_1-plane that starts at $x_1 = 0$, it goes along the line segment L_k, just before hitting the branched point $x_{1,k}$ it makes a small loop C_k counterclockwise around $x_{1,k}$, it returns back to the starting point along the line segment L_k, and then the loop continues to travel in a similar fashion along L_{k+1} except that this time we make a small loop C_{k+1}^{-1} clockwise around $x_{1,k+1}$. In other words $A'_k = L_{k+1}^{-1} \circ C_{k+1}^{-1} \circ L_{k+1} \circ L_k^{-1} \circ C_k \circ L_k$. Note that A'_k lifts to two loops $A_{k,a}$, $a \in \mathbb{Z}_2$ on M_μ, where the starting point of A'_k lifts to $x_{2,k,a} := \mu^{\frac{1}{2}}(-1)^a$. The cycles $A_{k,a}$ satisfy the following relations $A_{k,0} = -A_{k,1}$ and $\sum\limits_{k=0}^{N} A_{k,a} = 0$. Let us asume $a \in \mathbb{Z}_2 \setminus \{0\}$ and $k \in \mathbb{Z}_{N+1} \setminus \{0\}$, then we get N loops whose homology classes, as we will see later on, represent a basis of $H_1(M_\mu; \mathbb{Z})$.

Let us compute the periods of the holomorphic forms

$$\phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg} = -\frac{x_1^{i-1} dx_1}{2x_2}$$

along the cycles $A_{k,a}$. The paths L_k and C_k can be parametrized as follows:

$$L_k:\ x_1 = \eta_{N+1}^k \mu^{\frac{1}{N+1}} t^{\frac{1}{N+1}}, \qquad 0 \leqslant t \leqslant \left(1 - \frac{\epsilon}{\mu^{\frac{1}{N+1}}}\right)^{N+1},$$

$$C_k:\ x_1 = \eta_{N+1}^k \mu^{\frac{1}{N+1}} + \epsilon e^{\mathbf{i}\theta}, \qquad \frac{2k-N-1}{N+1}\pi \leqslant \theta \leqslant \frac{2k+N+1}{N+1}\pi.$$

The integrals along the lifts of C_k contribute to the period integral terms of order $O(\epsilon^{\frac{1}{2}})$. These terms vanish in the limit $\epsilon \to 0$. The periods that we want to compute are independent of ϵ for homotopy reasons. Therefore, by passing to the limit $\epsilon \to 0$ we get

$$\int_{A_{k,a}} \phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg} = (1 - (-1))\left(\int_{L_{k,a}} - \int_{L_{k+1,a}}\right)\frac{-x_1^{i-1} dx_1}{2x_2}$$

$$= \left(\int_{L_{k+1,a}} - \int_{L_{k,a}} \right) \frac{x_1^{i-1} dx_1}{x_2}.$$

The integrals along $L_{k,a}$ can be expressed in terms of Euler's Beta function $B(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$,

$$\int_{L_{k,a}} \frac{x_1^{i-1} dx_1}{x_2} = (-1)^a \int_0^1 \frac{\eta_{N+1}^{ki} \mu^{\frac{i}{N+1}} t^{\frac{i}{N+1}-1} dt}{(N+1)\mu^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} = (-1)^a \frac{\eta_{N+1}^{ki} \mu^{\frac{i}{N+1}-\frac{1}{2}}}{N+1} B\left(\frac{i}{N+1}, \frac{1}{2} \right).$$

Let $\alpha_{k,a} = \Sigma A_{k,a}$ be the suspension. Recalling formula (2.1) and using that

$$\int_0^\lambda (\lambda - \mu)^{1/2} \mu^a d\mu = \lambda^{a+3/2} B(a+1, 3/2), \tag{2.4}$$

we get

$$\left(I_{\alpha_{k,a}}^{(-1)}(\lambda), \phi_i \right) := \frac{1}{2\pi} \int_{\alpha_{k,a}} \phi_i \frac{\omega}{df} = \frac{1}{\pi} \partial_\lambda \int_0^\lambda (\lambda - \mu)^{\frac{1}{2}} \int_{A_{k,a}} \phi_i(x_1, x_2) \frac{dx_1 dx_2}{dg} d\mu$$

$$= (-1)^a \frac{\eta_{N+1}^{ki} (\eta_{N+1}^i - 1)}{2i} \lambda^{\frac{i}{N+1}}.$$

Recalling the formulas for the residue pairing we get

$$I_{\alpha_{k,a}}^{(-1)}(\lambda) = 4h \sum_{i=1}^N (I_{\alpha_{k,a}}^{(-1)}(\lambda), \phi_{h-i}) \phi_i.$$

By definition $\theta(\phi_i) = \left(\frac{1}{2} - \frac{i}{N+1} \right) \phi_i$. Therefore, using (1.1), we get

$$\Psi(\alpha_{k,a}) = (-1)^a 2 \sum_{i=1}^N \eta_{N+1}^{-ki} \left(\eta_{N+1}^{-i} - 1 \right) \Gamma\left(1 - \frac{i}{N+1} \right) \phi_i. \tag{2.5}$$

Let us point out that formula (2.2) yields the following formulas for the classical monodromy operator

$$\sigma(\alpha_{k,a}) = -\alpha_{k+1,a+1} = \alpha_{k+1,a}, \qquad k \in \mathbb{Z}_{N+1}, \qquad a \in \mathbb{Z}_2.$$

The intersection pairing takes the form

$$(\alpha_{k,1}|\alpha_{l,1}) = \frac{1}{\pi} (\Psi(\alpha_{k,1}), \cos(\pi\theta)\Psi(\alpha_{l,1}))$$

$$= \frac{2}{h} \sum_{i=1}^N \eta_{N+1}^{(l-k)i} \left(1 - \cos\left(\frac{2i\pi}{h} \right) \right) = 2\delta_{k,l} - \delta_{l-k,1} - \delta_{l-k,N},$$

where $k, l \in \mathbb{Z}_{N+1}$ and the Kronecker delta is also on $\mathbb{Z}_{N+1}$. Note that $(\alpha_{k,1}|\alpha_{k,1}) = 2$, so $\alpha_{k,1}$ is a vanishing cycle. The determinant of the intersection pairing in the basis $\{\alpha_{k,1}\}$, that is, the determinant of the matrix $(\alpha_{k,1}|\alpha_{l,1})_{k,l=1}^N$ is $N+1$, which coincides with the determinant of the Cartan matrix of the simple Lie algebra of type A_N. Therefore, $\{\alpha_{k,1}\}$ is a $\mathbb{Z}$-basis of the root lattice, that is, $H_2(V_\lambda; \mathbb{Z})$ and hence their images $\Psi(\alpha_{k,1})$ (see formula (2.5)) give a basis for the image of the Milnor lattice in H_f.

2.4 D_N-singularity

Let us fix the following basis of H_f:

$$\phi_i(x_1, x_2) = \begin{cases} x_2^{i-1} & \text{if } 1 \le i \le N - 1, \\ 2x_1 & \text{if } i = N. \end{cases}$$

The residue pairing takes the form

$$(\phi_i, \phi_j) = \frac{1}{2h}\delta_{i+j,N}, \quad 1 \le i, j \le N - 1, \qquad (\phi_i, \phi_N) = -\delta_{i,N}, \quad 1 \le i \le N,$$

where $h = 2N - 2$ is the Coxeter number. The Riemann surface M_μ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by the equation $x_1^2 x_2 + x_2^{N-1} = \mu$. The projection $(x_1, x_2) \mapsto x_2$ defines a degree 2 branched covering $M_\mu \to \mathbb{C}^*$, with branching points $x_{2,k} = \mu^{\frac{1}{N-1}}\eta^{2k}$ ($1 \le k \le N-1$), where $\eta = e^{2\pi i/h}$. Let A_k' be a simple loop in $\mathbb{C}^*$ around the line segment $L_k := [0, x_{2,k}]$, that is, A_k' is a loop starting at a point on the line segment L_k sufficiently close to 0, it goes along the line segment L_k, just before hitting the branch point $x_{2,k}$ it makes a small loop C_k around it and it returns back to the starting point along the line segment L_k, and finally it makes a small loop C_0 around 0. Clearly, the loop $A_k' = C_0 \circ L_k^{-1} \circ C_k \circ L_k$ lifts to a loop A_k in M_μ. Let us compute the periods of the holomorphic forms

$$\phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg} = \begin{cases} \dfrac{x_2^{i-1} dx_2}{2x_1 x_2} & \text{if } 1 \le i \le N - 1, \\ \dfrac{dx_2}{x_2} & \text{if } i = N, \end{cases}$$

along the cycle A_k. If $i = N$, then the period integral is just $2\pi i$. Suppose that $1 \le i \le N - 1$. Let us parametrize A_k' as follows:

$$C_0: \ x_2 = \epsilon e^{i\theta}, \qquad 0 \le \theta \le 2\pi,$$

$$L_k: \ x_2 = \mu^{\frac{1}{N-1}}\eta^{2k}t, \qquad \epsilon\mu^{-\frac{1}{N-1}} \le t \le 1 - \epsilon\mu^{-\frac{1}{N-1}},$$

$$C_k: \ x_2 = \mu^{\frac{1}{N-1}}\eta^{2k} + \epsilon e^{i\theta}, \qquad 0 \le \theta \le 2\pi.$$

The integrals along the lifts of C_0 and C_k contribute to the period integral terms of orders respectively $O(\epsilon^{i-1/2})$ and $O(\epsilon^{1/2})$. These terms vanish in the limit $\epsilon \to 0$. The two lifts of L_k, before and after going around the branch point $x_{2,k}$, have parametrizations, such that,

$$x_2 = \mu^{\frac{1}{N-1}}\eta^{2k}t, \qquad x_1 x_2 = \sqrt{(\mu - x_2^{N-1})x_2} = \eta^k \mu^{1/2+1/h}(1 - t^{N-1})^{1/2}t^{1/2},$$

where t varies from 0 to 1, and

$$x_2 = \mu^{\frac{1}{N-1}}\eta^{2k}t, \qquad x_1 x_2 = -\sqrt{(\mu - x_2^{N-1})x_2} = -\eta^k \mu^{1/2+1/h}(1 - t^{N-1})^{1/2}t^{1/2},$$

where t varies from 1 to 0. Now it is clear that the period integral, after passing to the limit $\epsilon \to 0$, takes the form

$$\int_{A_k} \phi_i \frac{dx_1 dx_2}{dg} = \mu^{\frac{m_i}{h} - \frac{1}{2}}\eta^{m_i k}\int_0^1 t^{i-\frac{3}{2}}(1 - t^{N-1})^{-1/2}dt,$$

where $m_i := 2i - 1$ ($1 \le i \le N - 1$). The above integral can be computed as follows,

$$\int_0^1 t^{i-3/2}(1 - t^{N-1})^{-1/2}dt = \frac{1}{N-1}\int_0^1 s^{(2i-1)\frac{1}{h}-1}(1 - s)^{-\frac{1}{2}}ds = \frac{1}{N-1}B\left(\frac{m_i}{h}, \frac{1}{2}\right).$$

We get the following formulas:

$$\int_{A_k} \phi_i \frac{dx_1 dx_2}{dg} = \begin{cases} \dfrac{1}{N-1} \mu^{\frac{m_i}{h} - \frac{1}{2}} \eta^{m_i k} B\left(\dfrac{m_i}{h}, \dfrac{1}{2}\right), & \text{if } 1 \le i \le N-1, \\ 2\pi\mathbf{i}, & \text{if } i = N. \end{cases}$$

Let $\alpha_k = \Sigma A_k$ be the suspension. Recalling formula (2.1) and using (2.4), we get

$$\frac{1}{2\pi} \int_{\alpha_k} \phi_i \frac{\omega}{df} = \begin{cases} \dfrac{1}{h} \eta^{m_i k} \dfrac{\lambda^{m_i/h}}{m_i/h}, & \text{if } 1 \le i \le N-1, \\ 2\mathbf{i}\lambda^{1/2}, & \text{if } i = N. \end{cases}$$

Therefore,

$$I_{\alpha_k}^{(-1)}(\lambda) = 2 \sum_{i=1}^{N-1} \eta^{m_i k} \frac{\lambda^{m_i/h}}{m_i/h} \phi_{N-i} - 2\mathbf{i}\lambda^{1/2}\phi_N.$$

Note that $\theta(\phi_i) = \left(\frac{m_{N-i}}{h} - \frac{1}{2}\right)\phi_i$ for $1 \le i \le N-1$ and $\theta(\phi_N) = 0$. Therefore

$$\Psi(\alpha_k) = 2 \sum_{i=1}^{N-1} \eta^{m_i k} \Gamma(m_i/h)\phi_{N-i} - \mathbf{i}\Gamma(m_N/h)\phi_N, \tag{2.6}$$

where $m_N = N - 1$.

Remark 2.1. The numbers $\frac{m_i}{h} = \frac{2i-1}{h}$ ($1 \le i \le N-1$), $\frac{m_N}{h} = \frac{1}{2}$ are the Coxeter exponents.

Put

$$v_k = 2 \sum_{i=1}^{N-1} \eta^{m_i k} \Gamma(m_i/h)\phi_{N-i}, \qquad 1 \le k \le N-1,$$

and $v_N = \mathbf{i}\Gamma(m_N/h)\phi_N$.

Proposition 2.2. *The image of the Milnor lattice under the map Ψ is the lattice in H_f with $\mathbb{Z}$-basis*

$$\beta_1 = v_1 - v_2, \ldots, \beta_{N-1} = v_{N-1} - v_N, \qquad \beta_N = v_{N-1} + v_N.$$

Proof. Using formula (2.3), it is straightforward to check that $\{v_i\}_{1 \le i \le N}$ is an orthonormal basis of H_f with respect to the intersection pairing, that is, $(v_i|v_j) = \delta_{i,j}$. We have $\Psi(\alpha_k) = v_k - v_N$ and $\Psi(\sigma\alpha_k) - \Psi(\alpha_{k+1}) = 2v_N$. Therefore, β_i belongs to the image of the Milnor lattice. On the other hand, since $(\beta_i|\beta_i) = 2$, we get that β_i is the image of a vanishing cycle. Recalling the root system interpretation of the set of vanishing cycles, we get that β_i ($1 \le i \le N$) are simple roots and that the corresponding Dynkin diagram is the Dynkin diagram of type D_N. Since the Milnor lattice is spanned by the set of vanishing cycles, the claim of the proposition follows. ∎

2.5 E_6-singularity

Let us fix the following basis of H_f:

$$\phi_i = \begin{cases} x_2^{i-1} & \text{if } 1 \le i \le 3, \\ x_1 x_2^{i-4} & \text{if } 4 \le i \le 6. \end{cases}$$

The residue pairing takes the form

$$(\phi_i, \phi_j) = \frac{1}{2h}\delta_{i+j,7}, \qquad 1 \leqslant i, j \leqslant 6,$$

where $h = 12$ is the Coxeter number. The Riemann surface M_μ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by the equation $x_1^3 + x_2^4 = \mu$. The projection $(x_1, x_2) \mapsto x_2$ defines a degree 3 branched covering $M_\mu \to \mathbb{C}$, with branching points $x_{2,k} = \mu^{\frac{1}{4}}\mathbf{i}^k$, $k \in \mathbb{Z}_4$.

Let L_k ($k \in \mathbb{Z}_4$) be the line segment $\left[0, \mu^{\frac{1}{4}}\mathbf{i}^k\right]$. Let A'_k be a loop in the x_2-plane $\mathbb{C}$ going around the branch points $x_{2,k}$ and $x_{2,k+1}$ in the following way: the loop starts at 0, it goes along the line segment L_k, just before hitting the branch point $x_{2,k}$ it makes a small loop C_k counterclockwise around $x_{2,k}$, it returns back to the starting point along L_k; then the loop travels in a similar fashion along L_{k+1} except that this time we make a small loop C_{k+1}^{-1} in *clockwise* direction around $x_{2,k+1}$. Clearly, the loop $A'_k = L_{k+1}^{-1} \circ C_{k+1}^{-1} \circ L_{k+1} \circ L_k^{-1} \circ C_k \circ L_k$ lifts to three loops $A_{k,a}$, $a \in \mathbb{Z}_3$ in M_μ, depending on how we choose the lift of the base point, i.e., the x_1-coordinate of the lift of the base point of A'_k could take the following values: $x_{1,a} = \mu^{\frac{1}{3}}\eta_3^a$, where $\eta_3 := e^{\frac{2\pi\mathbf{i}}{3}}$.

Let us consider the loops $A_{k,a}$ with $a \in \mathbb{Z}_3 \setminus \{0\}$ and $k \in \mathbb{Z}_4 \setminus \{0\}$. Let us compute the periods of the holomorphic forms

$$\phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg} = \begin{cases} \dfrac{x_2^{i-1}dx_2}{3x_1^2} & \text{if } 1 \leqslant i \leqslant 3, \\[3mm] \dfrac{x_2^{i-4}dx_2}{3x_1} & \text{if } 4 \leqslant i \leqslant 6, \end{cases}$$

along the cycles $A_{k,a}$. As a byproduct of our computation we will get that the homology classes of these 6 loops form a basis of $H_1(M_\mu; \mathbb{Z})$. Let us parametrize A'_k as follows:

$$L_k: \quad x_2 = \mathbf{i}^k \mu^{\frac{1}{4}}t^{\frac{1}{4}}, \qquad 0 \leqslant t \leqslant \left(1 - \frac{\epsilon}{\mu^{\frac{1}{4}}}\right)^4,$$

$$C_k: \quad x_2 = \mathbf{i}^k \mu^{\frac{1}{4}} + \epsilon e^{\mathbf{i}\theta}, \qquad \frac{k-2}{2}\pi \leqslant \theta \leqslant \frac{k+2}{2}\pi.$$

The integrals along the lifts of C_k contribute to the period integral terms of orders

$$\begin{cases} O\!\left(\epsilon^{\frac{1}{3}}\right) & \text{if } 1 \leqslant i \leqslant 3, \\[2mm] O\!\left(\epsilon^{\frac{2}{3}}\right) & \text{if } 4 \leqslant i \leqslant 6. \end{cases}$$

These terms vanish in the limit $\epsilon \to 0$. Therefore, under this limit, the periods of the holomorphic forms

$$\int_{A_{k,a}} \phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg} = \begin{cases} (1 - \eta_3^{-2})\left(\displaystyle\int_{L_{k,a}} - \int_{L_{k+1,a}}\right)\dfrac{\phi_i dx_2}{3x_1^2} & \text{if } 1 \leqslant i \leqslant 3, \\[5mm] (1 - \eta_3^{-1})\left(\displaystyle\int_{L_{k,a}} - \int_{L_{k+1,a}}\right)\dfrac{\phi_i dx_2}{3x_1^2} & \text{if } 4 \leqslant i \leqslant 6, \end{cases}$$

where $\eta_3 := e^{\frac{2\pi\mathbf{i}}{3}}$ and the integral

$$\int_{L_{k,a}} \frac{\phi_i dx_2}{3x_1^2} = \begin{cases} \displaystyle\int_0^1 \dfrac{\mathbf{i}^{ki}\mu^{\frac{i}{4}}t^{\frac{i}{4}-1}dt}{12\mu^{\frac{2}{3}}(1-t)^{\frac{2}{3}}\eta_3^{2a}} = \dfrac{\mathbf{i}^{ki}}{12}\eta_3^a \mu^{\frac{i}{4}-\frac{2}{3}}B\left(\dfrac{i}{4}, \dfrac{1}{3}\right) & \text{if } 1 \leqslant i \leqslant 3, \\[5mm] \displaystyle\int_0^1 \dfrac{\mathbf{i}^{k(i-3)}\mu^{\frac{i-3}{4}}t^{\frac{i-3}{4}-1}dt}{12\mu^{\frac{1}{3}}(1-t)^{\frac{1}{3}}\eta_3^a} = \dfrac{\mathbf{i}^{k(i-3)}}{12}\eta_3^{2a}\mu^{\frac{i-3}{4}-\frac{1}{3}}B\left(\dfrac{i-3}{4}, \dfrac{2}{3}\right) & \text{if } 4 \leqslant i \leqslant 6. \end{cases}$$

Then,

$$\int_{A_{k,a}} \phi_i(x_1, x_2) \frac{dx_1 dx_2}{dg}$$

$$= \begin{cases} \left(1 - \eta_3^{-2}\right)\left(1 - \mathbf{i}^i\right) \dfrac{\mathbf{i}^{ki}}{12} \eta_3^a \mu^{\frac{i}{4} - \frac{2}{3}} B\left(\dfrac{i}{4}, \dfrac{1}{3}\right) & \text{if } 1 \leqslant i \leqslant 3, \\[3mm] \left(1 - \eta_3^{-1}\right)\left(1 - \mathbf{i}^{i-3}\right) \dfrac{\mathbf{i}^{k(i-3)}}{12} \eta_3^{2a} \mu^{\frac{i-3}{4} - \frac{1}{3}} B\left(\dfrac{i-3}{4}, \dfrac{2}{3}\right) & \text{if } 4 \leqslant i \leqslant 6. \end{cases}$$

Let $\alpha_{k,a} = \Sigma A_{k,a}$ be the suspension. Recalling formula (2.1) and using (2.4)

$$\left(I_{\alpha_{k,a}}^{(-1)}(\lambda), \phi_i\right) := \frac{1}{2\pi} \int_{\alpha_{k,a}} \phi_i \frac{\omega}{df} = \frac{1}{\pi} \partial_\lambda \int_0^\lambda (\lambda - \mu)^{\frac{1}{2}} \int_{A_{k,a}} \phi_i(x_1, x_2) \frac{dx_1 dx_2}{dg} d\mu$$

$$= \begin{cases} \dfrac{\sqrt{3}}{12\pi} \eta_3^a e^{-\frac{\pi}{6}\mathbf{i}} \lambda^{\frac{i}{4} - \frac{1}{6}} \dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{i}{4}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{i}{4} + \frac{5}{6}\right)} \mathbf{i}^{ki} \left(1 - \mathbf{i}^i\right) & \text{if } 1 \leqslant i \leqslant 3, \\[3mm] \dfrac{\sqrt{3}}{12\pi} \eta_3^{2a} e^{\frac{\pi}{6}\mathbf{i}} \lambda^{\frac{i-3}{4} + \frac{1}{6}} \dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{i-3}{4}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{i-3}{4} + \frac{7}{6}\right)} \mathbf{i}^{k(i-3)} \left(1 - \mathbf{i}^{i-3}\right) & \text{if } 4 \leqslant i \leqslant 6. \end{cases}$$

Recalling the formulas for the residue pairing in the basis $\{\phi_i\}$ we get

$$I_{\alpha_{k,a}}^{(-1)}(\lambda) = 2h \sum_{i=1}^{6} \left(I_{\alpha_{k,a}}^{(-1)}(\lambda), \phi_{7-i}\right) \phi_i.$$

Recalling formula (1.1) and using that by definition

$$\theta(\phi_i) = \begin{cases} \left(\dfrac{2}{3} - \dfrac{i}{4}\right) \phi_i & \text{if } 1 \leqslant i \leqslant 3, \\[3mm] \left(\dfrac{1}{3} - \dfrac{i-3}{4}\right) \phi_i & \text{if } 4 \leqslant i \leqslant 6, \end{cases}$$

we get

$$\Psi(\alpha_{k,a}) = \sqrt{\frac{3}{\pi}} \sum_{i=1}^{3} e^{\frac{\pi}{6}\mathbf{i}} \eta_3^{2a} \Gamma\left(1 - \frac{i}{4}\right) \Gamma\left(\frac{2}{3}\right) \mathbf{i}^{-ki} \left(1 - \mathbf{i}^{-i}\right) \phi_i$$

$$+ \sqrt{\frac{3}{\pi}} \sum_{i=4}^{6} e^{-\frac{\pi}{6}\mathbf{i}} \eta_3^a \Gamma\left(1 - \frac{i-3}{4}\right) \Gamma\left(\frac{1}{3}\right) \mathbf{i}^{k(3-i)} \left(1 - \mathbf{i}^{3-i}\right) \phi_i. \tag{2.7}$$

Let us also point out that by using formula (2.2), we get the following formulas for the classical monodromy operator:

$$\sigma(\alpha_{k,a}) = -\alpha_{k+1,a+1}, \qquad k \in \mathbb{Z}_4, \qquad a \in \mathbb{Z}_3.$$

Recalling formula (2.3), we get that the intersection pairing

$$(\alpha_{k,a} | \alpha_{l,b}) = \frac{1}{\pi} (\Psi(\alpha_{k,a}), \cos(\pi\theta)\Psi(\alpha_{l,b}))$$

$$= \frac{1}{8} \sum_{i=1}^{3} \left(\eta_3^{b-a} \mathbf{i}^{(l-k)i} + \eta_3^{a-b} \mathbf{i}^{(k-l)i}\right) \frac{\cos\left(\left(\frac{i}{4} - \frac{2}{3}\right)\pi\right)}{\sin\left(\frac{i}{4}\pi\right)\sin\left(\frac{\pi}{3}\right)} \left(2 - \mathbf{i}^{-i} - \mathbf{i}^i\right)$$

$$= \sum_{i=1}^{3} \cos\left(\frac{2}{3}(b-a)\pi + \frac{i}{2}(l-k)\pi\right) \frac{\cos\left(\left(\frac{i}{4} - \frac{2}{3}\right)\pi\right)}{\sin\left(\frac{\pi}{3}\right)} \sin\left(\frac{i}{4}\pi\right).$$

Let us identify $\mathbb{Z}_3 \setminus \{0\} = \{1, 2\}$ and $\mathbb{Z}_4 \setminus \{0\} = \{1, 2, 3\}$. Every $1 \leq a' \leq 6$ can be written uniquely in the form $a' = 3(a-1)+k$, where $1 \leq a \leq 2$ and $1 \leq k \leq 3$. Let us define $\alpha_{a'} := \alpha_{k,a}$. The intersection pairings $(\alpha_{a'}|\alpha_{b'})$ are straightforward to compute using the formula from above. We get that $(\alpha_{a'}|\alpha_{b'})$ coincides with the (a', b')-entry of the following matrix:

$$
\begin{pmatrix}
2 & -1 & 0 & -1 & 0 & 0 \\
-1 & 2 & -1 & 1 & -1 & 0 \\
0 & -1 & 2 & 0 & 1 & -1 \\
-1 & 1 & 0 & 2 & -1 & 0 \\
0 & -1 & 1 & -1 & 2 & -1 \\
0 & 0 & -1 & 0 & -1 & 2
\end{pmatrix}.
$$

The above matrix has determinant 3. Therefore, the set $\{\alpha_{k,a} \,|\, 1 \leq a \leq 2, 1 \leq k \leq 3\}$ is a set of linearly independent vanishing cycles. Since the set of all vanishing cycles is a root system of type E_6 and the determinant of the Cartan matrix of a root system of E_6 is also 3, we get that the $\{\alpha_{k,a}\}$ is a set of simple roots. In particular, it is a $\mathbb{Z}$-basis of the Milnor lattice.

2.6 E_7-singularity

Let us fix the following basis of H_f:

$$
\phi_i = \begin{cases}
x_1^{i-1} & \text{if } 1 \leqslant i \leqslant 3, \\
x_2 x_1^{i-4} & \text{if } 4 \leqslant i \leqslant 6, \\
x_2^2 & \text{if } i = 7.
\end{cases}
$$

The residue pairing takes the form

$$
(\phi_i, \phi_j) = \begin{cases}
\frac{1}{h} \delta_{i+j,7}, & 1 \leqslant i, j \leqslant 6, \\
-\frac{1}{6}, & i = j = 7, \\
0, & \text{otherwise},
\end{cases}
$$

where $h = 18$ is the Coxeter number. The Riemann surface M_μ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by the equation $x_1^3 + x_1 x_2^3 = \mu$. The projection $(x_1, x_2) \mapsto x_1$ defines a degree 3 branched covering $M_\mu \to \mathbb{C}^*$, with branching points $x_{1,k} = \mu^{\frac{1}{3}} \eta_3^k$ $(0 \leqslant k \leqslant 2)$, where $\eta_3 = e^{\frac{2}{3}\pi \mathbf{i}}$.

The method of constructing loops in M_μ is similar to that of D_N-singularity. Let A'_k be a simple loop in $\mathbb{C}^*$ around the line segment $L_k := [0, x_{1,k}]$, that is, A'_k is a loop starting at a point $\tilde{\epsilon} \eta_3^k$ $(0 < \tilde{\epsilon} \ll 1)$ on the line segment L_k sufficiently close to 0, it goes along the line segment L_k, just before hitting the branch point $x_{1,k}$ it makes a small loop C_k counterclockwise around it, it returns back to the starting point along the line segment L_k, and finally it makes a small loop C_0 counterclockwise around 0. Clearly, the loop $A'_k = C_0 \circ L_k^{-1} \circ C_k \circ L_k$ lifts to a loop $A_{k,a}$, $a = 0, 1, 2$ in M_μ, where a indicates the lift of the base point, that is, the base point is lifted to $(x_{1,k,a} = \tilde{\epsilon} \eta_3^k, x_{2,k,a} = \left(\frac{\mu - \tilde{\epsilon}^3}{\tilde{\epsilon}}\right)^{\frac{1}{3}} \eta_3^{a - \frac{k}{3}})$. We will compute the period integrals along $A_{k,a}$ for $0 \leq k, a \leq 2$. As a biproduct of our computation we will get that the following set of 7 loops $\{A_{0,1}, A_{1,1}, A_{2,1}, A_{0,2}, A_{1,2}, A_{2,2}, A_{0,0}\}$ represents a basis of $H_1(M_\mu; \mathbb{Z})$.

Let us compute the periods of the holomorphic forms

$$
\phi_i(x_1, x_2) \frac{dx_1 dx_2}{dg} = \begin{cases}
-\dfrac{x_1^{i-2} dx_1}{3x_2^2} & \text{if } 1 \leq i \leq 3, \\
-\dfrac{x_1^{i-5}}{3x_2} dx_1 & \text{if } 4 \leq i \leq 6, \\
-\dfrac{dx_1}{3x_1} & \text{if } i = 7,
\end{cases}
$$

along the cycle $A_{k,a}$. Let us parametrize A'_k as follows:

$$C_0: \ x_1 = \tilde{\epsilon}e^{\mathbf{i}\theta}, \qquad \frac{2k}{3}\pi \le \theta \le \frac{2k+6}{3}\pi,$$

$$L_k: \ x_1 = \eta_3^k \mu^{\frac{1}{3}} t^{\frac{1}{3}}, \qquad \frac{\epsilon^3}{\mu} \leqslant t \leqslant \left(1 - \frac{\epsilon}{\mu^{\frac{1}{3}}}\right)^3,$$

$$C_k: \ x_1 = \eta_3^k \mu^{\frac{1}{3}} + \epsilon e^{\mathbf{i}\theta}, \qquad \frac{2k-3}{3}\pi \leqslant \theta \leqslant \frac{2k+3}{3}\pi.$$

The integrals along the lifts of C_0 and C_k contribute to the period integral terms of orders respectively

$$\begin{cases} O\big(\tilde{\epsilon}^{i-1+\frac{2}{3}}\big) \quad \text{and} \quad O\big(\epsilon^{1-\frac{2}{3}}\big) & \text{if } 1 \le i \le 3, \\ O\big(\tilde{\epsilon}^{i-4+\frac{1}{3}}\big) \quad \text{and} \quad O\big(\epsilon^{1-\frac{1}{3}}\big) & \text{if } 4 \le i \le 6, \\ O\big(\tilde{\epsilon}^0\big) \quad \text{and} \quad O\big(\epsilon^1\big) & \text{if } i = 7. \end{cases}$$

In the limit $\tilde{\epsilon}, \epsilon \to 0$ all integrals along the loops C_0 and C_k vanish except for the integral along C_0 when $i = 7$. The latter however is straightforward to compute. Therefore, after passing to the limit $\epsilon, \tilde{\epsilon} \to 0$, we get

$$\int_{A_{k,a}} \phi_i(x_1, x_2) \frac{dx_1 dx_2}{dg} = \begin{cases} \big(1 - \eta_3^{-2}\big) \displaystyle\int_{L_{k,a}} \frac{-\phi_i dx_1}{3x_1 x_2^2} & \text{if } 1 \le i \le 3, \\ \big(1 - \eta_3^{-1}\big) \displaystyle\int_{L_{k,a}} \frac{-\phi_i dx_1}{3x_1 x_2^2} & \text{if } 4 \le i \le 6, \\ -\dfrac{2\pi\mathbf{i}}{3} & \text{if } i = 7, \end{cases}$$

where $\eta_3 := e^{\frac{2\pi\mathbf{i}}{3}}$ and the integral

$$\int_{L_{k,a}} \frac{-\phi_i dx_1}{3x_1 x_2^2} = \begin{cases} \displaystyle\int_0^1 \frac{-\eta_3^{k(i-\frac{1}{3})-2a}\mu^{\frac{1}{3}(i-\frac{7}{3})}t^{\frac{1}{3}(i-\frac{10}{3})}dt}{9(1-t)^{\frac{2}{3}}} \\ \quad = -\dfrac{\eta_3^{k(i-\frac{1}{3})-2a}\mu^{\frac{1}{3}(i-\frac{7}{3})}}{9} B\left(\dfrac{i}{3} - \dfrac{1}{9}, \dfrac{1}{3}\right) & \text{if } 1 \leqslant i \leqslant 3, \\ \displaystyle\int_0^1 \frac{-\eta_3^{k(i-4+\frac{1}{3})-a}\mu^{\frac{1}{3}(i-4-\frac{2}{3})}t^{\frac{1}{3}(i-4-\frac{8}{3})}dt}{9(1-t)^{\frac{1}{3}}} \\ \quad = -\dfrac{\eta_3^{k(i-4+\frac{1}{3})-a}\mu^{\frac{1}{3}(i-4-\frac{2}{3})}}{9} B\left(\dfrac{i-4}{3} + \dfrac{1}{9}, \dfrac{2}{3}\right) & \text{if } 4 \leqslant i \leqslant 6. \end{cases}$$

Let $\alpha_{k,a} = \Sigma A_{k,a}$ be the suspension. Recalling formula (2.1) and using (2.4)

$$\big(I_{\alpha_{k,a}}^{(-1)}(\lambda), \phi_i\big) := \frac{1}{2\pi} \int_{\alpha_{k,a}} \phi_i \frac{\omega}{df} = \frac{1}{\pi}\partial_\lambda \int_0^\lambda (\lambda - \mu)^{\frac{1}{2}} \int_{A_{k,a}} \phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg}d\mu$$

$$= \begin{cases} -\dfrac{\sqrt{3}}{9\pi}\eta_3^{k(i-\frac{1}{3})-2a} e^{-\frac{\pi}{6}\mathbf{i}} \lambda^{\frac{i}{3}-\frac{7}{9}+\frac{1}{2}} \dfrac{\Gamma(\frac{3}{2})\Gamma(\frac{i}{3}-\frac{1}{9})\Gamma(\frac{1}{3})}{\Gamma(\frac{i}{3}-\frac{7}{9}+\frac{3}{2})} & \text{if } 1 \leqslant i \leqslant 3, \\ -\dfrac{\sqrt{3}}{9\pi}\eta_3^{k(i-4+\frac{1}{3})-a} e^{\frac{\pi}{6}\mathbf{i}} \lambda^{\frac{i-4}{3}-\frac{2}{9}+\frac{1}{2}} \dfrac{\Gamma(\frac{3}{2})\Gamma(\frac{i-4}{3}+\frac{1}{9})\Gamma(\frac{2}{3})}{\Gamma(\frac{i-4}{3}+\frac{7}{9}+\frac{1}{2})} & \text{if } 4 \leqslant i \leqslant 6, \\ -\dfrac{2}{3}\mathbf{i}\lambda^{\frac{1}{2}} & \text{if } i = 7. \end{cases}$$

Recalling the formulas for the residue pairing in the basis $\{\phi_i\}$ we get

$$I_{\alpha_{k,a}}^{(-1)}(\lambda) = h \sum_{i=1}^{6} \left(I_{\alpha_{k,a}}^{(-1)}(\lambda), \phi_{7-i} \right) \phi_i + 4\mathbf{i}\lambda^{\frac{1}{2}}\phi_7.$$

Recalling formula (1.1) and using that

$$\theta(\phi_i) = \begin{cases} \left(\dfrac{4}{9} - \dfrac{i-1}{3} \right) \phi_i & \text{if } 1 \leqslant i \leqslant 3, \\[2mm] \left(\dfrac{2}{9} - \dfrac{i-4}{3} \right) \phi_i & \text{if } 4 \leqslant i \leqslant 6, \\[2mm] 0 & \text{if } i = 7, \end{cases}$$

we get

$$\Psi(\alpha_{k,a}) = -\sqrt{\frac{3}{\pi}} \sum_{i=1}^{3} e^{\frac{\pi}{6}\mathbf{i}} \eta_3^{k(\frac{1}{3}-i)-a} \Gamma\left(\frac{3-i}{3} + \frac{1}{9} \right) \Gamma\left(\frac{2}{3} \right) \phi_i$$

$$- \sqrt{\frac{3}{\pi}} \sum_{i=4}^{6} e^{-\frac{\pi}{6}\mathbf{i}} \eta_3^{k(4-i-\frac{1}{3})-2a} \Gamma\left(\frac{7-i}{3} - \frac{1}{9} \right) \Gamma\left(\frac{1}{3} \right) \phi_i + 2\sqrt{-\pi}\phi_7. \tag{2.8}$$

Recalling formula (2.2), we get the following formulas for the classical monodromy operator:

$$\sigma(\alpha_{k,a}) = -\alpha_{k+1,a+1}, \qquad k, a \in \mathbb{Z}_3.$$

Using formula (2.3), we get that the intersection pairing

$$(\alpha_{k,a} | \alpha_{l,b}) = \frac{1}{\pi} (\Psi(\alpha_{k,a}), \cos(\pi\theta)\Psi(\alpha_{l,b}))$$

$$= \frac{2}{3} + \frac{1}{6} \sum_{i=1}^{3} \left(\eta_3^{b-a+(l-k)(i-\frac{1}{3})} + \eta_3^{a-b+(k-l)(i-\frac{1}{3})} \right) \frac{\cos\left(\left(\frac{i}{3} - \frac{7}{9} \right)\pi \right)}{\sin\left(\left(\frac{i}{3} - \frac{1}{9} \right)\pi \right) \sin\left(\frac{\pi}{3} \right)}$$

$$= \frac{2}{3} + \frac{1}{3} \sum_{i=1}^{3} \cos\left(\frac{2\pi}{3} \left(a - b + (k-l)\left(i - \frac{1}{3} \right) \right) \right) \frac{\cos\left(\left(\frac{i}{3} - \frac{7}{9} \right)\pi \right)}{\sin\left(\left(\frac{i}{3} - \frac{1}{9} \right)\pi \right) \sin\left(\frac{\pi}{3} \right)}.$$

Let us identify $\mathbb{Z}_3 = \{0,1,2\}$. Every $1 \leq a' \leq 7$ can be written uniquely in the form $a' = 3(a-1) + k + 1$, $1 \leq a \leq 3$, $0 \leq k \leq 2$. Put $\alpha_{a'} := \alpha_{k,\bar{a}}$, where $0 \leq \bar{a} \leq 2$ is the remainder of a modulo 3. Using the above formula, we get that the intersection pairing in the basis $\{\alpha_{a'}\}_{1 \leq a' \leq 7}$ takes the form

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The above matrix has determinant 2. Since the determinant of the Cartan matrix of the root system of type E_7 is also 2, the conclusion is the same as in the case of E_6-singularity, that is, the cycles $(\alpha_1, \ldots, \alpha_7) = (\alpha_{0,1}, \alpha_{1,1}, \alpha_{2,1}, \alpha_{0,2}, \alpha_{1,2}, \alpha_{2,2}, \alpha_{0,0})$ form a $\mathbb{Z}$-basis of the Milnor lattice and hence their images under Ψ, computed by formula (2.8), give a $\mathbb{Z}$-basis for the image of the Milnor lattice in H_f.

2.7 E_8-singularity

Let us fix the following basis of H_f:

$$\phi_i = \begin{cases} x_2^{i-1} & \text{if } 1 \leqslant i \leqslant 4, \\ x_1 x_2^{i-5} & \text{if } 5 \leqslant i \leqslant 8. \end{cases}$$

The residue pairing takes the form

$$(\phi_i, \phi_j) = \frac{1}{h}\delta_{i+j,9}, \qquad 1 \leqslant i, j \leqslant 8,$$

where $h = 30$ is the Coxeter number. The Riemann surface M_μ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by equation $x_1^3 + x_2^5 = \mu$. The projection $(x_1, x_2) \mapsto x_2$ defines a degree 3 branched covering $M_\mu \to \mathbb{C}$, with branching points $x_{2,k} = \mu^{\frac{1}{5}}\eta_5^k$, $k \in \mathbb{Z}_5$, where $\eta_5 = e^{\frac{2}{5}\pi\mathbf{i}}$.

The method for constructing loops in M_μ is almost the same as that for E_6-singularity. Let us omit the similar narration, i.e., we define the loops $A_{k,a}$ in the same way, except that now $a \in \mathbb{Z}_3 \setminus \{0\}$ and $k \in \mathbb{Z}_5 \setminus \{0\}$. We will see that the homology classes of these 8 loops form a $\mathbb{Z}$-basis of $H_1(M_\mu; \mathbb{Z})$. Let us compute the periods of the holomorphic forms

$$\phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg} = \begin{cases} \dfrac{x_2^{i-1} dx_2}{3x_1^2} & \text{if } 1 \leqslant i \leqslant 4, \\[2mm] \dfrac{x_2^{i-5} dx_2}{3x_1} & \text{if } 5 \leqslant i \leqslant 8, \end{cases}$$

along the cycle $A_{k,a}$. Let us parametrize A_k' as follows:

$$L_k: \quad x_2 = \eta_5^k \mu^{\frac{1}{5}} t^{\frac{1}{5}}, \qquad 0 \leqslant t \leqslant \left(1 - \frac{\epsilon}{\mu^{\frac{1}{5}}}\right)^5,$$

$$C_k: \quad x_2 = \eta_5^k \mu^{\frac{1}{5}} + \epsilon e^{\mathbf{i}\theta}, \qquad \frac{2k-5}{5}\pi \leqslant \theta \leqslant \frac{2k+5}{5}\pi.$$

The integrals along the lifts of C_k contribute to the period integral terms of orders $O\left(\epsilon^{\frac{1}{3}}\right)$ if $1 \leqslant i \leqslant 3$ and $O\left(\epsilon^{\frac{2}{3}}\right)$ if $4 \leqslant i \leqslant 6$. These terms vanish in the limit $\epsilon \to 0$. Therefore, under this limit, the periods of the holomorphic forms

$$\int_{A_{k,a}} \phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg} = \begin{cases} (1 - \eta_3^{-2})\left(\displaystyle\int_{L_{k,a}} - \int_{L_{k+1,a}}\right)\dfrac{\phi_i dx_2}{3x_1^2} & \text{if } 1 \leqslant i \leqslant 4, \\[4mm] (1 - \eta_3^{-1})\left(\displaystyle\int_{L_{k,a}} - \int_{L_{k+1,a}}\right)\dfrac{\phi_i dx_2}{3x_1^2} & \text{if } 5 \leqslant i \leqslant 8, \end{cases}$$

where $\eta_3 := e^{\frac{2\pi\mathbf{i}}{3}}$ and the integral

$$\int_{L_{k,a}} \frac{\phi_i dx_2}{3x_1^2} = \begin{cases} \displaystyle\int_0^1 \dfrac{\eta_5^{ki}\mu^{\frac{i}{5}}t^{\frac{i}{5}-1}dt}{15\mu^{\frac{2}{3}}(1-t)^{\frac{2}{3}}\eta_3^{2a}} = \dfrac{\eta_5^{ki}}{15}\eta_3^a \mu^{\frac{i}{5}-\frac{2}{3}}B\left(\dfrac{i}{5}, \dfrac{1}{3}\right) & \text{if } 1 \leqslant i \leqslant 4, \\[4mm] \displaystyle\int_0^1 \dfrac{\eta_5^{k(i-4)}\mu^{\frac{i-4}{5}}t^{\frac{i-4}{5}-1}dt}{15\mu^{\frac{1}{3}}(1-t)^{\frac{1}{3}}\eta_3^a} = \dfrac{\eta_5^{k(i-4)}}{15}\eta_3^{2a}\mu^{\frac{i-4}{5}-\frac{1}{3}}B\left(\dfrac{i-4}{5}, \dfrac{2}{3}\right) & \text{if } 5 \leqslant i \leqslant 8. \end{cases}$$

Then,

$$\int_{A_{k,a}} \phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg}$$

$$= \begin{cases} \left(1 - \eta_3^{-2}\right)\left(1 - \eta_5^i\right)\dfrac{\eta_5^{ki}}{15}\eta_3^a \mu^{\frac{i}{5} - \frac{2}{3}} B\left(\dfrac{i}{5}, \dfrac{1}{3}\right) & \text{if } 1 \leqslant i \leqslant 4, \\[2ex] \left(1 - \eta_3^{-1}\right)\left(1 - \eta_5^{i-4}\right)\dfrac{\eta_5^{k(i-4)}}{15}\eta_3^{2a}\mu^{\frac{i-4}{5} - \frac{1}{3}} B\left(\dfrac{i-4}{5}, \dfrac{2}{3}\right) & \text{if } 5 \leqslant i \leqslant 8. \end{cases}$$

Let $\alpha_{k,a} = \Sigma A_{k,a}$ be the suspension. Recalling formula (2.1) and using (2.4)

$$\left(I_{\alpha_{k,a}}^{(-1)}(\lambda), \phi_i\right) := \frac{1}{2\pi}\int_{\alpha_{k,a}} \phi_i \frac{\omega}{df} = \frac{1}{\pi}\partial_\lambda \int_0^\lambda (\lambda - \mu)^{\frac{1}{2}} \int_{A_{k,a}} \phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg}d\mu$$

$$= \begin{cases} \dfrac{\sqrt{3}}{15\pi}\eta_3^a e^{-\frac{\pi}{6}\mathbf{i}}\lambda^{\frac{i}{5} - \frac{1}{6}}\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{i}{5}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{i}{5} + \frac{5}{6}\right)}\eta_5^{ki}\left(1 - \eta_5^i\right) & \text{if } 1 \leqslant i \leqslant 4, \\[2ex] \dfrac{\sqrt{3}}{15\pi}\eta_3^{2a} e^{\frac{\pi}{6}\mathbf{i}}\lambda^{\frac{i-4}{5} + \frac{1}{6}}\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{i-4}{5}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{i-4}{5} + \frac{7}{6}\right)}\eta_5^{k(i-4)}\left(1 - \eta_5^{i-4}\right) & \text{if } 5 \leqslant i \leqslant 8. \end{cases}$$

Recalling the formulas for residue pairing, we get

$$I_{\alpha_{k,a}}^{(-1)}(\lambda) = h\sum_{i=1}^{8}\left(I_{\alpha_{k,a}}^{(-1)}(\lambda), \phi_{9-i}\right)\phi_i.$$

By definition,

$$\theta(\phi_i) = \begin{cases} \left(\dfrac{2}{3} - \dfrac{i}{5}\right)\phi_i, & \text{if } 1 \leqslant i \leqslant 4, \\[2ex] \left(\dfrac{1}{3} - \dfrac{i-4}{5}\right)\phi_i, & \text{if } 5 \leqslant i \leqslant 8. \end{cases}$$

Therefore, recalling formula (1.1), we get

$$\Psi(\alpha_{k,a}) = \sqrt{\frac{3}{\pi}}\sum_{i=1}^{4} e^{\frac{\pi}{6}\mathbf{i}}\eta_3^{2a}\Gamma\left(1 - \frac{i}{5}\right)\Gamma\left(\frac{2}{3}\right)\eta_5^{-ki}\left(1 - \eta_5^{-i}\right)\phi_i$$

$$+ \sqrt{\frac{3}{\pi}}\sum_{i=5}^{8} e^{-\frac{\pi}{6}\mathbf{i}}\eta_3^a\Gamma\left(1 - \frac{i-4}{5}\right)\Gamma\left(\frac{1}{3}\right)\eta_5^{k(4-i)}\left(1 - \eta_5^{4-i}\right)\phi_i. \tag{2.9}$$

Recalling formula (2.2), we get that the following formulas for the classical monodromy operator:

$$\sigma(\alpha_{k,a}) = -\alpha_{k+1,a+1}, \qquad k \in \mathbb{Z}_5, \qquad a \in \mathbb{Z}_3.$$

Recalling formula (2.3), the intersection pairing

$$(\alpha_{k,a}|\alpha_{l,b}) = \frac{1}{\pi}\left(\Psi(\alpha_{k,a}), \cos(\pi\theta)\Psi(\alpha_{l,b})\right)$$

$$= \frac{1}{10}\sum_{i=1}^{4}\left(\eta_3^{b-a}\eta_5^{(l-k)i} + \eta_3^{a-b}\eta_5^{(k-l)i}\right)\frac{\cos\left(\left(\frac{i}{5} - \frac{2}{3}\right)\pi\right)}{\sin\left(\frac{i}{5}\pi\right)\sin\left(\frac{\pi}{3}\right)}\left(2 - \eta_5^{-i} - \eta_5^i\right)$$

$$= \frac{4}{5}\sum_{i=1}^{4}\cos\left(\frac{2}{3}(b-a)\pi + \frac{2i}{5}(l-k)\pi\right)\frac{\cos\left(\left(\frac{i}{5} - \frac{2}{3}\right)\pi\right)}{\sin\left(\frac{\pi}{3}\right)}\sin\left(\frac{i}{5}\pi\right).$$

Let us identify $\mathbb{Z}_3 \setminus \{0\} = \{1, 2\}$ and $\mathbb{Z}_5 \setminus \{0\} = \{1, 2, 3, 4\}$. Every $1 \leq a' \leq 8$ can be written uniquely in the form $a' = 4(a-1) + k$, where $1 \leq a \leq 2$ and $1 \leq k \leq 4$. Put $\alpha_{a'} := \alpha_{k,a}$. Then

the intersection matrix $(\alpha_{a'}|\alpha_{b'})$ takes the following form:

$$
\begin{pmatrix}
2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 1 & -1 \\
-1 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 2
\end{pmatrix}.
$$

The above matrix has determinant 1. Since the determinant of the Cartan matrix of the root system of type E_8 is also 1, the conclusion is the same as in the previous cases.

3   K-theoretic interpretation

The goal of this section is to prove Theorem 1.3.

3.1   Fermat cases

Let us compute explicitly the map ch_Γ for $f(x) = f^T(x) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ with $a_3 = 2$. In fact, our computation works for arbitrary a_3 as well, except for one small technical detail, that is, we will prove that the group $K_{G^T}^{-1}(V_1^T)$ is torsion free. This fact should be true for any positive integer a_3, but the argument that we give works only if $a_3 = 2$. The group

$$
G^T = \left\{ g = (g_1, g_2, g_3) \in (\mathbb{C}^*) \,|\, g_1^{a_1} = g_2^{a_2} = g_3^{a_3} = 1 \right\}.
$$

If $g \in G^T$ is such that $I = \{i \,|\, g_i = 1\}$ is a non-empty set, then it is easy to see that the map $x = (x_1, x_2, x_3) \mapsto \left(x_i^{a_i}\right)_{i \in I}$ induces isomorphisms $\mathrm{Fix}_g\left(\mathbb{C}^3\right)/G^T \cong \mathbb{C}^I$ and $\mathrm{Fix}_g\left(V_1^T\right)/G^T \cong H_I$, where $H_I \subset \mathbb{C}^I$ is the hyperplane $\sum_{i \in I} y_i = 1$. Since the pair $\left(\mathbb{C}^I, H_I\right)$ is contractible the groups

$$
K^0\left(\mathrm{Fix}_g\left(\mathbb{C}^3\right)/G^T, \mathrm{Fix}_g\left(V_1^T\right)/G^T\right) = H^*\left(\mathrm{Fix}_g\left(\mathbb{C}^3\right)/G^T, \mathrm{Fix}_g\left(V_1^T\right)/G^T\right) = 0.
$$

If $g \in G^T$ is such that $g_i \neq 1$ for all i, then $\mathrm{Fix}_g\left(\mathbb{C}^3\right) = \{0\}$ and $\mathrm{Fix}_g\left(V_1^T\right) = \varnothing$. Note that the number of such g is $N = (a_1 - 1)(a_2 - 1)(a_3 - 1)$, that is, the multiplicity of the singularity corresponding to the polynomial f.

Lemma 3.1. *The group $K_{G^T}^{-1}\left(V_1^T\right)$ is torsion free.*

Let us postpone the proof of this lemma until Section 3.4. Note that $K_{G^T}^{-1}\left(V_1^T\right) \otimes \mathbb{C} = 0$. Therefore, according to the above Lemma 3.1, we have $K_{G^T}^{-1}\left(V_1^T\right) = 0$. The long exact sequence of the pair $\left(\mathbb{C}^3, V_1^T\right)$ yields the following exact sequence

$$
0 \longrightarrow K_{G^T}^0\left(\mathbb{C}^3, V_1^T\right) \longrightarrow K_{G^T}^0\left(\mathbb{C}^3\right) \longrightarrow K_{G^T}^0\left(V_1^T\right).
$$

On the other hand, $K_{G^T}^0\left(\mathbb{C}^3\right)$ coincides with the representation ring of G^T, that is,

$$
K_{G^T}^0\left(\mathbb{C}^3\right) = \mathbb{Z}[L_1, L_2, L_3]/\left(L_1^{a_1} - 1, L_2^{a_2} - 1, L_3^{a_3} - 1\right),
$$

where $L_i = \mathbb{C}^3 \times \mathbb{C}$ is the trivial bundle with G^T-action $g \cdot (x, \lambda) := (gx, g_i \lambda)$. Note that $T\mathbb{C}^3 \cong L_1 + L_2 + L_3$ in the category of G^T-equivariant bundles. We claim that

$$
K_{G^T}^0\left(\mathbb{C}^3, V_1^T\right) = (L_1 - 1)(L_2 - 1)(L_3 - 1)\mathbb{Z}[L_1, L_2, L_3]/\left(L_1^{a_1} - 1, L_2^{a_2} - 1, L_3^{a_3} - 1\right). \quad (3.1)
$$

Indeed, note that $s_i(x) = (x, f_{x_i})$ is a G^T-equivariant section of L_i^{-1}. The Koszul complex corresponding to the sequence (s_1, s_2, s_3) has the form

$$L_1 L_2 L_3 \longrightarrow \bigoplus_{1 \leq i < j \leq 3} L_i L_j \longrightarrow \bigoplus_{1 \leq i \leq 3} L_i \longrightarrow \mathbb{C},$$

where $\mathbb{C}$ is the trivial bundle with trivial G^T-action. The sequence (s_1, s_2, s_3) is regular, so the corresponding Koszul complex is a resolution of the structure sheaf of the zero locus $\{s_1 = s_2 = s_3 = 0\}$. The zero locus is $\{0\}$ and since $0 \notin V_1^T$ the restriction of the Koszul complex to V_1^T is exact, i.e., the Koszul complex represents an element of $K_{G^T}^0(\mathbb{C}^3, V_1^T)$. This proves that the RHS of (3.1) is a $\mathbb{Z}$-submodule of the LHS. Note that both the LHS and the RHS of (3.1) are free $\mathbb{Z}$-modules of rank N. Therefore, the quotient of LHS by RHS is a finite Abelian group. In order to prove that the quotient is 0, it is sufficient to prove that if $g \in K_{G^T}^0(\mathbb{C}^3)$ and mg belongs to the RHS of (3.1) for some integer m, then g belongs to the RHS of (3.1) too. The proof is straightforward so we leave it as an exercise.

Let us fix the following basis of $K_{G^T}^0(\mathbb{C}^3, V_1^T)$:

$$A_{m_1, m_2, m_3} := L_1^{m_1} L_2^{m_2} L_3^{m_3} (L_1 - 1)(L_2 - 1)(L_3 - 1), \qquad 0 \leq m_i \leq a_i - 2.$$

Let $e_{k_1, k_2, k_3} = 1 \in H^0\big(\operatorname{Fix}_g(\mathbb{C}^3)/G, \operatorname{Fix}_g(V_1^T)/G\big)$, where $g = \big(e^{2\pi i k_1/a_1}, e^{2\pi i k_2/a_2}, e^{2\pi i k_3/a_3}\big)$. We get

$$\operatorname{ch}_\Gamma(A_{m_1, m_2, m_3}) = \frac{1}{2\pi} \sum_{k_1=1}^{a_1-1} \sum_{k_2=1}^{a_2-1} \sum_{k_3=1}^{a_3-1} \prod_{i=1}^{3} \left(\Gamma\left(1 - \frac{k_i}{a_i}\right) e^{-2\pi i k_i m_i/a_i} \big(e^{-2\pi i k_i/a_i} - 1\big) \right) e_{k_1, k_2, k_3},$$

where the ingredients of the above formula are computed as follows. Since $\operatorname{Fix}_g(\mathbb{C}^3) = \{0\}$ and the action of g on $L_i|_{\operatorname{Fix}_g(\mathbb{C}^3)}$ is given by multiplication by $e^{2\pi i k_i/a_i}$ we get

$$\widehat{\Gamma}(L_i)|_{\operatorname{Fix}_g(\mathbb{C}^3)/G^T} = \Gamma\left(1 - \frac{k_i}{a_i}\right) \qquad \text{and}$$

$$\iota^* \widetilde{\operatorname{ch}}(L_i)|_{\operatorname{Fix}_g(\mathbb{C}^3)/G^T} = \widetilde{\operatorname{ch}}\big(L_i^{-1}\big)|_{\operatorname{Fix}_g(\mathbb{C}^3)/G^T} = e^{-2\pi i k_i/a_i},$$

where we used that $\iota^*(L_i) = L_i^{-1}$. The orbifold tangent bundle $\big[T\mathbb{C}^3/G^T\big] = L_1 + L_2 + L_3$ so its Γ-class is $\prod_{i=1}^{3} \widehat{\Gamma}(L_i)$, while $\iota^* \widetilde{\operatorname{ch}}|_{\operatorname{Fix}_g(\mathbb{C}^3)/G^T}$ is a ring homomorphism, so the computation of its value on A_{m_1, m_2, m_3} amounts to the substitution $L_i \mapsto e^{-2\pi i k_i/a_i}$. Let us specialize the above formula to the cases of A_N, E_6, and E_8 singularities. In the first case $a_1 = N + 1$, $a_2 = a_3 = 2$. The above formula takes the form

$$\operatorname{ch}_\Gamma(A_{m,0,0}) = 2 \sum_{k=1}^{N} \eta^{-km} \big(\eta^{-k} - 1\big) \Gamma\left(1 - \frac{k}{N+1}\right) e_{k,1,1}.$$

Comparing with (2.5), we get that if we define $\operatorname{mir}(\phi_i) = e_{i,1,1}$ ($1 \leq i \leq N$), then the images of Ψ and ch_Γ will coincide. The vanishing cycle $\alpha_{k,a}$ corresponds to $(-1)^a A_{k,0,0}$.

For the case of E_6 we have $a_1 = 3$, $a_2 = 4$, $a_3 = 2$. The formula takes the form

$$\operatorname{ch}_\Gamma(A_{m_1, m_2, 0}) = -\frac{1}{\sqrt{\pi}} \sum_{k_1=1}^{2} \sum_{k_2=1}^{3} \Gamma\left(1 - \frac{k_1}{3}\right) \Gamma\left(1 - \frac{k_2}{4}\right)$$

$$\times \eta_3^{-k_1 m_1} \eta_4^{-k_2 m_2} \big(\eta_3^{-k_1} - 1\big)\big(\eta_4^{-k_2} - 1\big) e_{k_1, k_2, 1},$$

where $\eta_3 = e^{2\pi\mathbf{i}/3}$ and $\eta_4 = e^{2\pi\mathbf{i}/4} = \mathbf{i}$. Note that $\eta_3^{-1} - 1 = -\sqrt{3}e^{\pi\mathbf{i}/6}$ and $\eta_3^{-2} - 1 = -\sqrt{3}e^{-\pi\mathbf{i}/6}$. Comparing with (2.7) we get that if we define

$$\mathrm{mir}(\phi_i) = \begin{cases} e_{1,i,1}, & \text{for } 1 \leq i \leq 3, \\ e_{2,i-3,1}, & \text{for } 4 \leq i \leq 6, \end{cases}$$

then the images of Ψ and ch_Γ will coincide. The vanishing cycle $\alpha_{k,a}$ corresponds to $-A_{a,k,0}$.

Suppose now that the singularity is of type E_8, that is, $a_1 = 3$, $a_2 = 5$, and $a_3 = 2$. The formula takes the form

$$\mathrm{ch}_\Gamma(A_{m_1,m_2,0}) = -\frac{1}{\sqrt{\pi}} \sum_{k_1=1}^{2} \sum_{k_2=1}^{4} \Gamma\left(1 - \frac{k_1}{3}\right) \Gamma\left(1 - \frac{k_2}{5}\right)$$
$$\times \eta_3^{-k_1 m_1} \eta_5^{-k_2 m_2} \left(\eta_3^{-k_1} - 1\right)\left(\eta_5^{-k_2} - 1\right) e_{k_1,k_2,1},$$

where $\eta_3 = e^{2\pi\mathbf{i}/3}$ and $\eta_5 = e^{2\pi\mathbf{i}/5}$. Comparing with formula (2.9) we get that if we define

$$\mathrm{mir}(\phi_i) = \begin{cases} e_{1,i,1}, & \text{for } 1 \leq i \leq 4, \\ e_{2,i-4,1}, & \text{for } 5 \leq i \leq 8, \end{cases}$$

then the images of Ψ and ch_Γ will coincide. The vanishing cycle $\alpha_{k,a}$ corresponds to $-A_{a,k,0}$.

3.2   Suspension isomorphism

Suppose that X is a finite CW-complex equipped with an action of a finite (or more generally compact Lie) group G. Let $\mu_2 = \{\pm 1\}$ and

$$\Sigma X = X \times \{-1\} \big\backslash {}^{X \times [-1,1]} \big/ {}_{X \times \{1\}}$$

be the suspension of X, where the double quotient simply means that the quotient is taken in two steps: first by, say, $X \times \{-1\}$ and then by $X \times \{1\}$. Note that $G \times \mu_2$ acts naturally on ΣX via $(g, \epsilon) \cdot (x, t) = (gx, \epsilon t)$ and that the 0-dimensional sphere $\mathbb{S}^0 := \Sigma X - X \times (-1, 1)$ is a $G \times \mu_2$-equivariant subcomplex of ΣX.

Let $L = [-1, 1] \times \mathbb{C}$ be the trivial μ_2-equivariant line bundle on the interval $I := [-1, 1]$, where the representation of μ_2 on $\mathbb{C}$ is given by $\epsilon \cdot \lambda = \epsilon\lambda$. It is an easy and amusing exercise to check that $K_{\mu_2}^0(I, \partial I) = \mathbb{Z}\,\ell$, where ℓ is the relative K-theoretic class of the complex $L \xrightarrow{t} \mathbb{C}$, where $\mathbb{C} = I \times \mathbb{C}$ is the trivial μ_2-equivariant line bundle corresponding to the trivial representation of μ_2 on $\mathbb{C}$ and the map is induced by $(t, \lambda) \mapsto (t, t\lambda)$.

Lemma 3.2. *The exterior tensor product by ℓ induces an isomorphism*

$$K_G^i(X) \cong K_{G \times \mu_2}^i(\Sigma X, \mathbb{S}^0).$$

Proof. By definition

$$K_{G \times \mu_2}^i(\Sigma X, \mathbb{S}^0) = \widetilde{K}_{G \times \mu_2}^i(\Sigma X / \mathbb{S}^0) = K_{G \times \mu_2}^i(X \times I, X \times \partial I) \cong K_G^i(X) \otimes K_{\mu_2}^0(I, \partial I),$$

where we used that $K_{\mu_2}^i(I, \partial I)$ is isomorphic to $\mathbb{Z}$ for i even and 0 for i odd, so the last isomorphism is given by the equivariant Künneth formula (see [15]).   ∎

Suppose now that $Y \subset X$ is a G-invariant CW-subcomplex of X. Using the long exact sequence of the triple $\mathbb{S}^0 \subset \Sigma Y \subset \Sigma X$ and Lemma 3.2, it is straightforward to prove the following corollary.

Corollary 3.3. *The exterior tensor product by ℓ induces an isomorphism*

$$K_G^i(X, Y) \cong K_{G \times \mu_2}^i(\Sigma X, \Sigma Y).$$

3.3 The relative K-ring for D_N-singularity

Let us return to the settings of D_N-singularity. We have $f^T(x) = x_1^2 + x_1 x_2^{N-1} + x_3^2$. The group of diagonal symmetries of f^T is

$$G^T = \big\{ t \in (\mathbb{C}^*)^3 \,|\, t_1^2 = t_1 t_2^{N-1} = t_3^2 = 1 \big\}.$$

Let $L_i = \mathbb{C}^3 \times \mathbb{C}$ be the G^T-equivariant line bundle for which the action of G^T on $\mathbb{C}$ is given by the character $G^T \to \mathbb{C}^*$, $(t_1, t_2, t_3) \mapsto t_i$. Let us introduce the following N complexes of G^T-equivariant vector bundles on $\mathbb{C}^3$:

$$E_i^\bullet: \quad L_1 L_2^{i-1} L_3 \xrightarrow{\;d_0\;} L_1 L_2^{i-1} \oplus L_2^{i-1} L_3 \xrightarrow{\;d_1\;} L_2^{i-1}, \qquad 1 \le i \le N-1,$$

where the differentials are defined by $d_0(x, \lambda) = (x, -x_3\lambda, x_1\lambda)$ and $d_1(x, \lambda_1, \lambda_3) = (x, x_1\lambda_1 + x_3\lambda_3)$,

$$E_N^\bullet: \quad L_3 \xrightarrow{\;d_0\;} \mathbb{C} \oplus L_3 \xrightarrow{\;d_1\;} \mathbb{C},$$

where $d_0(x, \lambda) = (x, -x_3\lambda, x_1^2\lambda)$ and $d_1(x, \lambda_1, \lambda_3) = (x, x_1^2\lambda_1 + x_3\lambda_3)$.

Proposition 3.4. *The relative K-ring $K^0_{G^T}(\mathbb{C}^3, V^T) \cong \mathbb{Z}^N$ and the complexes $E_i^\bullet$ $(1 \le i \le N)$ represent a $\mathbb{Z}$-basis.*

Proof. Note that the complex $E_i^\bullet$ $(1 \le i \le N-1)$ is a tensor product of L_2^{i-1}, $L_1 \xrightarrow{\;x_1\;} \mathbb{C}$, and $L_3 \xrightarrow{\;x_3\;} \mathbb{C}$ and that the complex $E_N^\bullet$ is a tensor product of $\mathbb{C} \xrightarrow{\;x_1^2\;} \mathbb{C}$ and $L_3 \xrightarrow{\;x_3\;} \mathbb{C}$. On the other hand, we have $G^T = A \times \mu_2$, where $A = \big\{ t \in (\mathbb{C}^*)^2 \,|\, t_1^2 = t_1 t_2^{N-1} = 1 \big\}$. Recalling Corollary 3.3 we get $K^0_{G^T}(\mathbb{C}^3, V^T) \cong K^0_A(\mathbb{C}^2, M)$, where $M = \big\{ x \in \mathbb{C}^2 \,|\, x_1^2 + x_1 x_2^{N-1} = 1 \big\}$. Slightly abusing the notation we denote by L_1 and L_2 the restriction of the vector bundles L_1 and L_2 to $\mathbb{C}^2$. Note that the operation tensor product by the complex $L_3 \xrightarrow{\;x_3\;} \mathbb{C}$ is precisely the exterior tensor product by the complex ℓ in the suspension isomorphism from Corollary 3.3. Therefore, it is sufficient to prove that the complexes $L_1 L_2^{i-1} \xrightarrow{\;x_1\;} L_2^{i-1}$ $(1 \le i \le N-1)$ and $\mathbb{C} \xrightarrow{\;x_1^2\;} \mathbb{C}$ represent a $\mathbb{Z}$-basis of $K^0_A(\mathbb{C}^2, M)$.

The long exact sequence of the pair $(\mathbb{C}^2, M)$ yields the following exact sequence:

$$0 \longrightarrow K^{-1}_A(M) \xrightarrow{\;\delta\;} K^0_A(\mathbb{C}^2, M) \xrightarrow{\;\rho\;} K^0_A(\mathbb{C}^2) \longrightarrow K^0_A(M),$$

where we used that $K^{-1}_A(\mathbb{C}^2) = 0$. We have

$$K^0_A(\mathbb{C}^2) = \mathbb{Z}[L_1, L_2]/\big\langle L_1^2 - 1, L_1 L_2^{N-1} - 1 \big\rangle,$$

where the RHS is the representation ring of A. Just like in the Fermat cases it is easy to prove that the image of ρ coincides with the ideal $(L_1 - 1)K^0_A(\mathbb{C}^2)$. Note that $\mathrm{Im}(\rho) \cong \mathbb{Z}^{N-1}$ and that $\rho(E_i^\bullet) = L_2^{i-1}(L_1 - 1)$ $(1 \le i \le N-1)$ is a $\mathbb{Z}$-basis. It remains only to prove that $K^{-1}_A(M) \cong \mathbb{Z}$ and that $\mathrm{Im}(\delta)$ is generated as a $\mathbb{Z}$-module by the complex $E_N^\bullet$.

Let us first prove that $K^{-1}_A(M) \cong \mathbb{Z}$. Let $\pi\colon M \to \mathbb{C}^*$ be the map $(x_1, x_2) \mapsto x_1^2$. The map π is a branched covering with only one branch point, that is, $1 \in \mathbb{C}^*$. The corresponding ramification points are $R = \{(-1, 0), (1, 0)\}$. Note that R is an A-invariant subset. The idea is to use the long exact sequence of the pair $(M, M \setminus R)$. The action of A on $M \setminus R$ is free, so we have

$$K^i_A(M \setminus R) = K^i((M \setminus R)/A) = K^i(\mathbb{C} \setminus \{0, 1\}).$$

Therefore $K_A^0(M \setminus R) \cong \mathbb{Z}$ and $K_A^{-1}(M \setminus R) \cong \mathbb{Z}^2$. The groups $K^i(M, M \setminus R)$ are also easy to compute. Let $U \subset M$ be a small A-invariant open neighborhood of R. Then by excision $K_A^i(M, M \setminus R) = K_A^i(U, U \setminus R)$. Note that the open neighborhood U can be identified with an open neighborhood of R in the normal bundle ν_R to R in M. Indeed, the normal bundle is trivial $\nu_R = R \times \mathbb{C}$ and a point in $(x_1, x_2) \in U$ satisfies $x_1 = \frac{1}{2}\left(-x_2^{N-1} \pm \sqrt{x_2^{2N-2} + 4}\right)$, so the map $U \to \nu_R$, $(x_1, x_2) \mapsto ((\pm 1, 0), x_2)$ identifies U with an open neighborhood of the zero section R in ν_R. Clearly, the pullback of ν_R to U is L_2 and the Thom class of ν_R is represented as an element of $K_A^0(\nu_R) = K_A^0(U, U \setminus R)$ by the complex $\mathbb{C} \xrightarrow{x_2} L_2$. According to Thom isomorphism $K_A^i(U, U \setminus R) \cong K_A^i(R)$. Note that R is an A-orbit, that is, $R = A/B$, where B is the cyclic subgroup of A generated by $(1, \eta^2)$. Therefore, $K_A^{-1}(R) = 0$ and $K_A^0(R)$ coincides with the representation ring of B. Since the Thom isomorphism is given by tensor product with the Thom class, we get

$$K_A^0(M, M \setminus R) = \bigoplus_{i=1}^{N-1} \mathbb{Z}\left[L_2^{i-1} \xrightarrow{x_2} L_2^i\right].$$

The long exact sequence of the pair $(M, M \setminus R)$ takes the form

$$0 \longrightarrow K_A^{-1}(M) \longrightarrow K_A^{-1}(M \setminus R) \xrightarrow{\delta} K_A^0(M, M \setminus R).$$

We already proved that $K_A^{-1}(M \setminus R) \cong K^{-1}(\mathbb{C}^2 \setminus \{0, 1\}) \cong \mathbb{Z}^2$. We will make use of the following explicit interpretation of the K-group $K_A^{-1}(\ )$. By definition, for any finite CW-complex X, we have $K_A^{-1}(X) = \widetilde{K}_A^0(\Sigma(X \sqcup \mathrm{pt}))$. Since the complement of X in $\Sigma(X \sqcup \mathrm{pt})$ is contractible, we can think of an element of $K_A^{-1}(X)$ as a representation of A on some vector space $\mathbb{C}^r$ and an A-equivariant isomorphism $\phi \colon X \times \mathbb{C}^r \to X \times \mathbb{C}^r$, that is, an A-equivariant morphism $X \to \mathrm{GL}_r(\mathbb{C})$. In our case the elements of $K_A^{-1}(M \setminus R)$ are obtained by pullback from $K^{-1}(\mathbb{C} \setminus \{0, 1\})$. The latter is generated by two elements that correspond, in the way described above, to the two maps $\mathbb{C} \setminus \{0, 1\} \to \mathbb{C}^*$, $t \mapsto t$ and $t \mapsto 1 - t$. Therefore, the group $K_A^{-1}(M \setminus R)$ is generated by the two elements that correspond to the two maps $M \setminus R \to \mathbb{C}^*$ defined by $(x_1, x_2) \mapsto x_1^2$ and $(x_1, x_2) \mapsto 1 - x_1^2 = x_1 x_2^{N-1}$. The connecting morphism δ can be described as follows. Given an A-equivariant isomorphism $\phi \colon (M \setminus R) \times \mathbb{C}^r \to (M \setminus R) \times \mathbb{C}^r$, then let us pick an extension to a vector bundle morphism $\widetilde{\phi} \colon M \times \mathbb{C}^r \to M \times \mathbb{C}^r$. The resulting complex clearly represents an element of $K_A^0(M, M \setminus R)$ and that is what $\delta(\phi)$ is. The extensions in our case are straightforward to construct. We get that

$$\mathrm{Im}(\delta) = \mathbb{Z}\left[\mathbb{C} \xrightarrow{x_1^2} \mathbb{C}\right] + \mathbb{Z}\left[\mathbb{C} \xrightarrow{x_1 x_2^{N-1}} \mathbb{C}\right].$$

Note however, that $x_1 \neq 0$ so x_1^2 defines an isomorphism, i.e., the first complex is 0 in $K_A^0(M, M \setminus R)$. In particular, the kernel of the connecting homomorphism δ is $\cong \mathbb{Z}$ and it is generated by the element in $K_A^{-1}(M \setminus R)$ corresponding to the map $M \setminus R \to \mathbb{C}^*$, $(x_1, x_2) \to x_1^2$. This map extends to M, so we get that $K_A^{-1}(M) \cong \mathbb{Z}$ with generator corresponding to the map $M \to \mathbb{C}^*$, $(x_1, x_2) \mapsto x_1^2$. Returning to the long exact sequence of the pair $(\mathbb{C}^2, M)$, we get that the connecting morphism $K_A^{-1}(M) \to K_A^0(\mathbb{C}^2, M)$ maps the generator of $K_A^{-1}(M)$ to the complex $\mathbb{C} \xrightarrow{x_1^2} \mathbb{C}$. This completes the proof of the proposition. ∎

3.4 Proof of Lemma 3.1

We follow the same strategy as in the proof of Proposition 3.4. Let us denote by $M \subset \mathbb{C}^2$ the Riemann surface defined by $x_1^{a_1} + x_2^{a_2} = 1$. Let $A = \{t \in (\mathbb{C}^*)^2 \mid t_1^{a_1} = t_2^{a_2} = 1\}$. Since

$K_{G^T}^{-1}(V_1^T) \cong K_A^{-1}(M)$, we get that it is sufficient to prove that $K_A^{-1}(M)$ is torsion free. Let $\pi\colon M \to \mathbb{C}$ be the map $(x_1, x_2) \mapsto x_1^{a_1}$. The map π is a branched covering with only one branching point, that is, $1 \in \mathbb{C}$. The corresponding ramification points are $R = \{(\xi, 0) \mid \xi^{a_1} = 1\}$. The torsion freeness can be deduced easily from the long exact sequence of the pair $(M, M \setminus R)$. The action of A on $M \setminus R$ is free, so we have

$$K_A^{-1}(M \setminus R) = K^{-1}((M \setminus R)/A) = K^{-1}(\mathbb{C} \setminus \{1\}) \cong \mathbb{Z}.$$

Using the Thom isomorphism for the normal bundle to R in M, we get $K_A^{-1}(M, M \setminus R) = K_A^{-1}(R)$. On the other hand, note that R is the orbit of A through the point $(1, 0) \in M$, we get $R = A/B$, where $B \subset A$ is the cyclic subgroup generated by $(1, \eta_{a_2})$, $\eta_{a_2} = e^{2\pi\mathbf{i}/a_2}$. Therefore, $K_A^{-1}(R) = K_A^{-1}(A/B) = K^{-1}(B) = 0$. Recalling the long exact sequence of the pair $(M, M \setminus R)$, we get

$$0 \longrightarrow K_A^{-1}(M) \longrightarrow K_A^{-1}(M \setminus R) \xrightarrow{\ \delta\ } K_A^0(M, M \setminus R). \tag{3.2}$$

We get that $K_A^{-1}(M)$ can be embedded as a subgroup of $K_A^{-1}(M \setminus R) \cong \mathbb{Z}$. The latter is torsion free, so $K_A^{-1}(M)$ must be also torsion free.

Remark 3.5. The above argument can be continued to give a direct proof of the fact that $K_A^{-1}(M) = 0$. Namely, using the Thom isomorphism, we can prove that the group $K_A^0(M, M \setminus R)$ is a free Abelian group of rank a_2 and that the complexes $\left[L_2^{i-1} \xrightarrow{\ x_2\ } L_2^i \right]$ $(1 \le i \le a_2)$ represent a $\mathbb{Z}$-basis. Moreover, the image of the connecting morphism δ in (3.2) can be computed explicitly as well, that is, it coincides with the sum of the above complexes. In particular, we get that δ is an injective map, and hence $K_A^{-1}(M) = 0$.

Remark 3.6. The long exact sequences of the pairs $(M, M \setminus R)$ and $(\mathbb{C}^2, M)$ can be computed explicitly, that is, both the groups and the differentials can be determined. This allows us to give an alternative proof of formula (3.1). We leave the details to the interested reader.

3.5 The relative K-ring for E_7-singularity

The argument from the previous section works also for E_7-singularity. Let us only state the result. The proof is completely analogous.

We have $f^T(x) = x_1^3 x_2 + x_2^3 + x_3^2$. The group of diagonal symmetries of f^T is

$$G^T = \{t \in (\mathbb{C}^*)^3 \mid t_1^3 t_2 = t_2^3 = t_3^2 = 1\}.$$

Let $L_i = \mathbb{C}^3 \times \mathbb{C}$ be the G^T-equivariant line bundle for which the action of G^T on $\mathbb{C}$ is given by the character $G^T \to \mathbb{C}^*$, $(t_1, t_2, t_3) \mapsto t_i$. Let us introduce the following 7 complexes of G^T-equivariant vector bundles on $\mathbb{C}^3$:

$$E_i^\bullet\colon\quad L_1^{i-1} L_3^{-1} \xrightarrow{\ d_0\ } L_1^{i-1} \oplus L_1^{i-1} L_2 L_3^{-1} \xrightarrow{\ d_1\ } L_1^{i-1} L_2, \qquad 1 \le i \le 6,$$

where the differentials are defined by $d_0(x, \lambda) = (x, -x_3\lambda, x_2\lambda)$ and $d_1(x, \lambda_2, \lambda_3) = (x, x_2\lambda_2 + x_3\lambda_3)$,

$$E_7^\bullet\colon\quad L_3 \xrightarrow{\ d_0\ } \mathbb{C} \oplus L_3 \xrightarrow{\ d_1\ } \mathbb{C},$$

where $d_0(x, \lambda) = (x, -x_3\lambda, x_2^3\lambda)$ and $d_1(x, \lambda_2, \lambda_3) = (x, x_2^3\lambda_2 + x_3\lambda_3)$.

Proposition 3.7. *Let $V^T = \{x \in \mathbb{C}^3 \mid f^T(x) = 1\}$. The relative K-ring $K_{G^T}^0(\mathbb{C}^3, V^T) \cong \mathbb{Z}^7$ and the complexes $E_i^\bullet$ $(1 \le i \le 7)$ represent a $\mathbb{Z}$-basis.*

3.6 Γ-integral structure for D_N-singularity

Let us compute $\mathrm{ch}_\Gamma(E_i^\bullet)$ for $1 \leq i \leq N$. After a straightforward computation we get that the relative cohomology group $H\big(\mathrm{Fix}_g\left(\mathbb{C}^3\right), \mathrm{Fix}_g\left(V^T\right)\big)^{G^T}$ is not zero only in the following two cases: 1) $g = (g_1, g_2, g_3)$ with $g_i \neq 1$ for all i and 2) $g = (1, 1, -1)$. For the first case, there are $N - 1$ elements, that is, $g = \left(-1, \eta^{2i-1}, -1\right)$ $(1 \leq i \leq N - 1)$ and the fixed point subsets are $\mathrm{Fix}_g\left(\mathbb{C}^3\right) = \{0\}$ and $\mathrm{Fix}_g\left(V^T\right) = \varnothing$. Therefore, $H\big(\mathrm{Fix}_g\left(\mathbb{C}^3\right), \mathrm{Fix}_g\left(V^T\right); \mathbb{C}\big)^{G^T} \cong \mathbb{C}$ is non-trivial only in degree 0 and we denote by $e_i := 1$ the unit of the cohomology group. For the second case, $\mathrm{Fix}_g\left(\mathbb{C}^3\right) = \mathbb{C}^2$ and $\mathrm{Fix}_g\left(V^T\right) = M = \left\{x_1^2 + x_1 x_2^{N-1} = 1\right\}$. The relative cohomology group $H^i\big(\mathbb{C}^2, M; \mathbb{C}\big)^{G^T} \cong H^{i-1}\big(M/G^T; \mathbb{C}\big)$ for $i > 0$ and $= 0$ for $i = 0$. As we already explained above $M/G^T = \mathbb{C}^*$, so the relative cohomology is non-zero only in degree 2, i.e., for $i = 2$. Since M is a Stein manifold, we can describe the relative cohomology in terms of the holomorphic de Rham complexes on $\mathbb{C}^2$ and M. Namely, consider the complex of Abelian groups

$$\Gamma\big(\mathbb{C}^2, \Omega_{\mathbb{C}^2}^\bullet\big)^{G^T} \oplus \Gamma\big(M, \Omega_M^{\bullet-1}\big)^{G^T}, \qquad d(\omega, \alpha) = (d\omega, \omega|_M - d\alpha). \tag{3.3}$$

A closed form (ω, α) in degree i, that is, $d(\omega, \alpha) = 0$, defines naturally a linear functional on the space of dimension i relative chains $\gamma \subset \mathbb{C}^2$ with $\partial\gamma \subset M$, that is,

$$\gamma \mapsto \int_\gamma \omega - \int_{\partial\gamma} \alpha.$$

Using the de Rham theorem for $\mathbb{C}^2$ and M, it is easy to prove that the above map induces an isomorphism between the i-th cohomology of the complex (3.3) and $H^i\big(\mathbb{C}^2, M; \mathbb{C}\big)^{G^T}$. Let us denote by $e_N \in H^2\big(\mathbb{C}^2, M; \mathbb{C}\big)$ the cohomology class corresponding to the form $\left(0, -\frac{1}{2\pi\mathbf{i}} dx_1/x_1\right)$.

Suppose now that $g = \left(-1, \eta^{2a-1}, -1\right)$, $1 \leq a \leq N - 1$. Let us compute the component of $\mathrm{ch}_\Gamma(E_i^\bullet)$ for $1 \leq i \leq N - 1$ in $H^0\big(\mathrm{Fix}_g\left(\mathbb{C}^3\right), \mathrm{Fix}_g\left(V^T\right); \mathbb{C}\big)^{G^T}$. Note that in this case we have an isomorphism $K^0\big(\mathrm{Fix}_g\left(\mathbb{C}^3\right), \mathrm{Fix}_g\left(V^T\right)\big) \cong K^0\big(\mathrm{Fix}_g\left(\mathbb{C}^3\right)\big)$. The image of $\iota^* \mathrm{Tr}(E_i^\bullet)$ is

$$\eta^{-(2a-1)(i-1)} L_2^{i-1}(L_1 L_3 - (-L_3 - L_1) + \mathbb{C}) = 4\eta^{-(2a-1)(i-1)}\mathbb{C},$$

where again we abused the notation by denoting by L_i the restriction of L_i to $\mathrm{Fix}_g\left(\mathbb{C}^3\right) = \{0\}$. The component of the Γ-class is

$$\Gamma(L_1 + L_2 + L_3) = \Gamma(1 - 1/2)\Gamma(1 - (2a - 1)/h)\Gamma(1 - 1/2)e_a,$$

where $h := 2N - 2$. Therefore, the component of $\mathrm{ch}_\Gamma(E_i^\bullet)$ is

$$2\eta^{-(2a-1)(i-1)}\Gamma(1 - m_a/h)e_a.$$

The component of $\mathrm{ch}_\Gamma(E_N)$ is clearly 0, because the image of the complex $E_N^\bullet$ in $K^0\big(\mathrm{Fix}_g\left(\mathbb{C}^3\right)\big)$ is 0.

Suppose that $g = (1, 1, -1)$. Let us compute the component of $\mathrm{ch}_\Gamma(E_i^\bullet)$ in

$$H^2\big(\mathrm{Fix}_g\left(\mathbb{C}^3\right), \mathrm{Fix}_g\left(V^T\right); \mathbb{C}\big)^{G^T} = H^2\big(\mathbb{C}^2, M\big)^{G^T} = \mathbb{C}e_N.$$

By definition the component of $\iota^* \mathrm{Tr}(E_i^\bullet)$ in $K^0\big(\mathbb{C}^2, M\big)^{G^T}$ is

$$-L_2^{i-1}\left[L_1 L_3 \xrightarrow{-x_1} L_3 \longrightarrow 0\right] + L_2^{i-1}\left[0 \longrightarrow L_1 \xrightarrow{x_1} \mathbb{C}\right],$$

where the above complexes are concentrated in degrees 0, 1, and 2 and the vector bundles L_i $(1 \leq i \leq 3)$ are trivial line bundles on $\mathbb{C}^2$. The second complex, as an element of $K^0(\mathbb{C}^2, M)$, is equivalent to the two-term complex $\mathbb{C} \xrightarrow{\bar{x}_1} \mathbb{C}$. Therefore, the component of $\iota^* \operatorname{Tr}(E_i^{\bullet})$ $(1 \leq i \leq N-1)$ takes the form

$$-\left[\mathbb{C} \xrightarrow{x_1} \mathbb{C}\right] + \left[\mathbb{C} \xrightarrow{\bar{x}_1} \mathbb{C}\right].$$

In order to compute the Chern character of the above complexes, we use the following commutative diagram:

$$
\begin{array}{ccc}
\widetilde{K}^0(\Sigma M) \cong \widetilde{K}^{-1}(M) & \xrightarrow{\cong} & K^0(\mathbb{C}^2, M) \\
\text{ch} \downarrow & & \downarrow \text{ch} \\
H^2(\Sigma M) \cong H^1(M) & \xrightarrow{\delta} & H^2(\mathbb{C}^2, M),
\end{array}
$$

where the horizontal arrows come from the long exact sequence of the pair $(\mathbb{C}^2, M)$ and the vertical arrows are isomorphisms. Under the isomorphism $\widetilde{K}^0(\Sigma M) \cong K^0(\mathbb{C}^2, M)$, the complex $\mathbb{C} \xrightarrow{x_1} \mathbb{C}$ corresponds to $P-1$, where P is a line bundle on ΣM obtained by gluing two trivial line bundles along M using the gluing function $M \to \mathbb{C}^*$, $(x_1, x_2) \mapsto x_1$. The first Chern class of P is easy to compute. If γ is a closed loop in M representing a cohomology class in $H_1(M)$, then $\Sigma\gamma$ is a sphere in $H_2(\Sigma M)$ and hence $P|_{\Sigma\gamma}$ is a line bundle on the sphere obtained from gluing two trivial line bundles on the two hemi-spheres along the equator γ using the map $\gamma \to \mathbb{C}^*$, $(x_1, x_2) \to x_1$. By definition $\langle c_1(P), \Sigma\gamma \rangle$ coincides with the degree of the map $\gamma \to \mathbb{S}^1$, $(x_1, x_2) \mapsto x_1/|x_1|$, that is,

$$\langle c_1(P), \Sigma\gamma \rangle = \frac{1}{2\pi \mathbf{i}} \int_{\gamma} \frac{dx_1}{x_1}.$$

In other words, under the suspension isomorphism, $c_1(P)$ coincides with the de Rham cohomology class of the form $\frac{1}{2\pi \mathbf{i}} dx_1/x_1$. Recalling the de Rham model for the relative cohomology group $H^2(\mathbb{C}^2, M)$, we get that $\delta(c_1(P)) = e_N$. Note that $c_1(P) = \operatorname{ch}(P-1)$, so $\operatorname{ch}\left(\mathbb{C} \xrightarrow{x_1} \mathbb{C}\right) = e_N$.

The vector bundle corresponding to the other complex $\mathbb{C} \xrightarrow{\bar{x}_1} \mathbb{C}$ is P^{-1}, so we get

$$\operatorname{ch}(\iota^* \operatorname{Tr}(E_i^{\bullet})) = -2e_N.$$

Hence

$$\operatorname{ch}_{\Gamma}(E_i^{\bullet})_g = \frac{1}{2\pi}\Gamma(1/2)(2\pi \mathbf{i})(-2e_N) = -2\mathbf{i}\Gamma(1/2)e_N,$$

where the index g is to remind us that this is the component corresponding to the fixed point set of $g = (1, 1, -1)$. The computation of $\operatorname{ch}_{\Gamma}(E_N^{\bullet})$ is the same, except that everywhere we have to replace the vector bundle P with P^2, so

$$\operatorname{ch}_{\Gamma}(E_N^{\bullet})_g = \frac{1}{2\pi}\Gamma(1/2)(2\pi \mathbf{i})(-4e_N) = -4\mathbf{i}\Gamma(1/2)e_N.$$

Combining our computations we get the following result:

$$\operatorname{ch}_{\Gamma}(E_i^{\bullet}) = 2\sum_{a=1}^{N-1} \eta^{-(2a-1)(i-1)}\Gamma(1-m_a/h)e_a - 2\mathbf{i}\Gamma(1/2)e_N$$

and $\operatorname{ch}_{\Gamma}(E_N^{\bullet}) = -4\mathbf{i}\Gamma(1/2)e_N$. Comparing with formula (2.6) we get that if we define $\operatorname{mir}(\phi_i) = e_i$ for $1 \leq i \leq N-1$ and $\operatorname{mir}(\phi_N) = 2e_N$, then the statement of Theorem 1.3 will hold. The vanishing cycle α_k $(1 \leq k \leq N-1)$ corresponds to the relative K-theoretic class of the complex $E_k^{\bullet}$.

3.7 Γ-integral structure for E_7-singularity

The computation in this case is similar to the case of D_N-singularity. Let us sketch only the main steps and leave the details as an exercise. The goal is to compute $\mathrm{ch}_\Gamma(E_l^\bullet)$ for $1 \leq l \leq 7$. After a straightforward computation we get that the relative cohomology group $H\big(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T)\big)^{G^T}$ is not zero only in the following two cases: 1) $g = (g_1, g_2, g_3)$ with $g_i \neq 1$ for all i and 2) $g = (1, 1, -1)$. Put $\eta = e^{2\pi\mathbf{i}/9}$ and $\eta_3 = e^{2\pi\mathbf{i}/3}$. For the first case, there are 6 elements, that is, $g = \big(\eta^{3i-r}, \eta_3^r, -1\big)$ ($1 \leq i \leq 3$, $1 \leq r \leq 2$) and the fixed-point subsets are $\mathrm{Fix}_g(\mathbb{C}^3) = \{0\}$ and $\mathrm{Fix}_g(V^T) = \varnothing$. Therefore, $H\big(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T); \mathbb{C}\big)^{G^T} \cong \mathbb{C}$ is non-trivial only in degree 0 and we denote by $e_{3i-r} \in H\big(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T); \mathbb{C}\big)^{G^T}$ the unit of the cohomology group. For the second case, $\mathrm{Fix}_g(\mathbb{C}^3) = \mathbb{C}^2$ and $\mathrm{Fix}_g(V^T) = M = \{x_1^3 x_2 + x_2^3 = 1\}$. The relative cohomology group $H^i(\mathbb{C}^2, M; \mathbb{C})^{G^T} \cong H^{i-1}(M/G^T; \mathbb{C})$ for $i > 0$ and $= 0$ for $i = 0$. Just like in the D_N-case, we have $M/G^T = \mathbb{C}^*$, so the relative cohomology is non-zero only in degree 2, i.e., for $i = 2$. Let us denote by $e_7 \in H^2(\mathbb{C}^2, M)^{G^T}$ the cohomology class corresponding to the differential form $\big(0, -\frac{1}{2\pi i} dx_2/x_2\big)$.

Suppose that $g = \big(\eta^{3i-r}, \eta_3^r, -1\big)$. We have

$$\iota^* \mathrm{Tr}(E_l^\bullet)_g = -2\big(1 - \eta_3^{-r}\big)\eta^{-(3i-r)(l-1)}\mathbb{C} \in K^0\big(\mathrm{Fix}_g(\mathbb{C}^3)\big)$$

and the component of the Γ-class of $\big[T\mathbb{C}^3/G^T\big]$ in $H\big(\mathrm{Fix}_g(\mathbb{C}^3)/G^T\big)$ is

$$\Gamma(1 - (3i - r)/9)\Gamma(1 - r/3)\Gamma(1/2).$$

Therefore

$$\mathrm{ch}_\Gamma(E_l^\bullet)_g = -\frac{1}{\sqrt{\pi}}\big(1 - \eta_3^{-r}\big)\eta^{-(3i-r)(l-1)}\Gamma(1 - (3i - r)/9)\Gamma(1 - r/3)e_{3i-r}, \qquad 1 \leq l \leq 6.$$

Note that $\mathrm{ch}_\Gamma(E_7^\bullet)_g = 0$.

Suppose now that $g = (1, 1, -1)$. Then we have

$$\iota^* \mathrm{Tr}(E_l^\bullet)_g = -\big[\mathbb{C} \xrightarrow{x_2} \mathbb{C} \longrightarrow 0\big] + \big[0 \longrightarrow \mathbb{C} \xrightarrow{x_2} \mathbb{C}\big]. \tag{3.4}$$

Under the isomorphism $\widetilde{K}^0(\Sigma M) \cong K^0(\mathbb{C}^2, M)$, the complex $\big[\mathbb{C} \xrightarrow{x_2} \mathbb{C}\big]$ corresponds to $P - 1$, where P is a vector bundle on ΣM obtained by gluing two trivial line bundles along M with gluing function $M \to \mathbb{C}^*$, $(x_1, x_2) \mapsto x_2$. Under the suspension isomorphism $H^2(\Sigma M) \cong H^1(M)$, we have that $\mathrm{ch}(P - 1) = c_1(P)$ is the cohomology class corresponding to the form $\frac{1}{2\pi\mathbf{i}} dx_2/x_2$. The latter, under the boundary isomorphism $H^1(M) \to H^2(\mathbb{C}^2, M)$ is mapped precisely to e_7, that is, $\mathrm{ch}\big(\big[\mathbb{C} \xrightarrow{x_2} \mathbb{C}\big]\big) = e_7$. The Chern character of the second complex in (3.4) is $-e_7$, so we get $\mathrm{ch}(\iota^* \mathrm{Tr}(E_l^\bullet))_g = -2e_7$. Hence

$$\mathrm{ch}_\Gamma(E_l^\bullet)_g = \frac{1}{2\pi}\Gamma(1/2)(2\pi\mathbf{i})(-2e_7) = -2\mathbf{i}\sqrt{\pi}e_7.$$

The computation of $\mathrm{ch}_\Gamma(E_7^\bullet)_g$ is the same as above except that we have to replace the bundle P with P^3, that is, we get $\mathrm{ch}_\Gamma(E_7^\bullet)_g = -6\mathbf{i}\sqrt{\pi}e_7$.

Collecting the results of our computations we get

$$\mathrm{ch}_\Gamma(E_l^\bullet) = -\sum_{r=1}^{2}\sum_{i=1}^{3}\frac{1}{\sqrt{\pi}}\big(1 - \eta_3^{-r}\big)\eta^{-(3i-r)(l-1)}\Gamma(1 - (3i - r)/9)\Gamma(1 - r/3)e_{3i-r} - 2\mathbf{i}\sqrt{\pi}e_7$$

for $1 \leq l \leq 6$ and $\mathrm{ch}_\Gamma(E_7^\bullet) = -6\mathbf{i}\sqrt{\pi}e_7$. Let us compare the above formula with (2.8). Note that $1 - \eta_3^{-1} = \sqrt{3}e^{\pi\mathbf{i}/6}$ and $1 - \eta_3^{-2} = \sqrt{3}e^{-\pi\mathbf{i}/6}$. We would like to find $k, a \in \mathbb{Z}_3$, such that, $\mathrm{mir}\circ\Psi(\alpha_{k,a}) = \mathrm{ch}_\Gamma(E_l^\bullet)$. Let us write $l - 1 = 3m + k$ for $0 \leq k \leq 2$, $0 \leq m \leq 1$. Then the above formula will hold if we choose $a = -m$ and define

$$\mathrm{mir}(\phi_i) = e_{3i-1}, \quad 1 \leq i \leq 3, \qquad \mathrm{mir}(\phi_{j+3}) = e_{3j-2}, \quad 1 \leq j \leq 3, \qquad \mathrm{mir}(\phi_7) = -e_7.$$

Note that $\mathrm{mir}\circ\Psi(\alpha_{k,0} + \alpha_{k,1} + \alpha_{k,2}) = \mathrm{ch}_\Gamma(E_7^\bullet)$. Using these formulas, we get immediately that the maps $\mathrm{mir}\circ\Psi$ and ch_Γ identify the Milnor lattice $H_2\big(f^{-1}(1);\mathbb{Z}\big)$ with the relative K-ring $K^0\big(\mathbb{C}^3, V^T\big)$. This completes the proof of Theorem 1.3.

Acknowledgements

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