On the Quantum K-Theory of the Quintic

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Abstract. Quantum K-theory of a smooth projective variety at genus zero is a collection of integers that can be assembled into a generating series $J(Q, q, t)$ that satisfies a system of linear differential equations with respect to t and q-difference equations with respect to Q. With some mild assumptions on the variety, it is known that the full theory can be reconstructed from its small J-function $J(Q, q, 0)$ which, in the case of Fano manifolds, is a vector-valued q-hypergeometric function. On the other hand, for the quintic 3-fold we formulate an explicit conjecture for the small J-function and its small linear q-difference equation expressed linearly in terms of the Gopakumar–Vafa invariants. Unlike the case of quantum knot invariants, and the case of Fano manifolds, the coefficients of the small linear q-difference equations are not Laurent polynomials, but rather analytic functions in two variables determined linearly by the Gopakumar–Vafa invariants of the quintic. Our conjecture for the small J-function agrees with a proposal of Jockers–Mayr.

Key words: quantum K-theory; quantum cohomology; quintic; Calabi–Yau manifolds; Gromov–Witten invariants; Gopakumar–Vafa invariants; q-difference equations; q-Frobenius method; J-function; reconstruction; gauged linear σ models; 3d-3d correspondence; Chern–Simons theory; q-holonomic functions

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1 Introduction

1.1 Quantum K-theory, the small *J*-function and its *q*-difference equation

The K-theoretic Gromov–Witten invariants of a compact Kähler manifold X (often omitted from the notation) is a collection of integers (see [27, p. 6])

$$\left\langle E_1 L^{k_1}, \dots, E_n L^{k_n} \right\rangle_{g,n,d} \tag{1.1}$$

defined for vector bundles $E_1, \dots, E_n$ on X and nonnegative integers $k_1, \dots, k_n$ as the holomorphic Euler characteristic of $\mathcal{O}^{\mathrm{vir}} \otimes \left(\otimes_{i=1}^n \mathrm{ev}_i^*(E_i) \otimes L_i^{k_1} \right)$ over the moduli space $\overline{\mathcal{M}}_{g,n}^{X,d}$ of genus g degree d stable maps to X with n marked points. Here, $L_1, \dots, L_n$ denote the line (orbi)bundles over $\overline{\mathcal{M}}_{g,n}^{X,d}$ formed by the cotangent lines to the curves at the respective marked points. A definition of these integers was given by Givental and Lee [22, 33]. These numerical invariants can be assembled into a generating series which at genus zero can be used to define an associative deformation of the product of the K-theory ring $K(X)$ of X.

There are several ways to assemble the integers (1.1) into generating series, and reconstruction theorems relate these generating series and often determine one from the other. This is reviewed

in Section 2.2. Our choice of generating series will be the so-called small J-function

$$J_X(Q,q,0) = (1-q)\Phi_0 + \sum_d \sum_\alpha \left\langle \frac{\Phi_\alpha}{1-qL} \right\rangle_{0,1,d} \Phi^\alpha Q^d \in K(X) \otimes \mathcal{K}_-(q)[[Q]] \qquad (1.2)$$

(with the notation of Section 2.1), which determines the genus 0 quantum K-theory X, i.e., the integers (1.1) [28, Theorem 1.1, Lemma 3.3] with $g = 0$, as well as the genus 0 permutation-equivariant quantum K-theory X [24] (when $K(X)$ is generated by line bundles).

The small J-function is a vector-valued function (taking values in the rational vector space $K(X)$) that obeys a system of linear q-difference equations [26, 27], giving rise to matrices $A_i(Q,q,0) \in K(X) \otimes \mathcal{K}_+(q)[[Q]]$, for $i = 1,\ldots,r$ which can also be used to reconstruct the genus 0 quantum K-theory of X [28, Lemma 3.3]. Concretely, for $X = \mathbb{CP}^N$, the small J-function is given by a q-hypergeometric formula [26, 27, 33]

$$J_{\mathbb{CP}^N}(Q,q,0) = (1-q) \sum_{d=0}^\infty \frac{Q^d}{((1-x)q;q)_d^{N+1}} \in K(\mathbb{CP}^N) \otimes \mathcal{K}_-(q)[[Q]], \qquad (1.3)$$

where $(z;q)_d = \prod_{j=0}^{d-1} \left(1 - q^j z\right)$ for $d \geq 0$, and

$$K(\mathbb{CP}^N) = \mathbb{Q}[x]/(x^{N+1})$$

is the K-theory ring with basis $\{1, x, \ldots, x^N\}$ where $1-x$ is the class of $\mathcal{O}(1)$.[1] The corresponding matrix $A(Q,q,0)$ of the vector-valued q-holonomic function $J(Q,q,0)$ is given by [28, Section 4.1]

$$A(Q,q,0) = I - \begin{pmatrix} 0 & 0 & \ldots & 0 & Q \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \qquad (1.4)$$

in the above basis of $K(\mathbb{CP}^N)$. It is remarkable that either (1.3) or (1.4) give the complete determination of all the integers (1.1) for $\mathbb{CP}^N$. Observe that the small J-function of $\mathbb{CP}^N$ is given by a vector-valued q-hypergeometric formula, which is always q-holonomic (as follows from Zeilberger et al. [35, 41, 43]), and as a result the entries of $A(Q,q,0)$ (as well as the coefficients of the small quantum product) are polynomials in Q and q. It turns out that the small J-function of Grassmanianns, flag varieties, homogeneous spaces and more generally Fano manifolds is q-hypergeometric as shown by many researchers; see, e.g., [5, 37, 38] and references therein. On the other hand, new phenomena are expected for the case of general Calabi–Yau manifolds, and particularly for the quintic. Our motivation to study the case of the quintic was two-fold, coming from numerical observations concerning coincidences of quantum K-theory counts and quantum cohomology counts (given below), as well as a comparison of the linear q-difference equations in quantum K-theory with those in Chern–Simons theory (such as the q-difference equation of the colored Jones polynomial of a knot [19]).

Our results give a relation between quantum K-theory and quantum cohomology of the quintic in two different limits, namely $q = 1$ (see Corollary 1.3) and $q = 0$ (see Corollary 1.5), and propose a linear expression of the small J-function of the quintic in terms of its Gopakumar–Vafa invariants (see Conjecture 1.1).

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[1]The K-theory ring is also written as [28, Section 4.1] $K(\mathbb{CP}^N) = \mathbb{Q}[P,P^{-1}]/((1-P)^{N+1})$ as the Grothendieck group of locally free sheaves on projective space, where $P = \mathcal{O}_{\mathbb{P}^N}(-1)$ in which case the small J-function takes the form $J_{\mathbb{CP}^N}(Q,q,0) = (1-q)\sum_{d=0}^\infty \frac{Q^d}{(Pq;q)_d^{N+1}}$.

1.2   The small J-function for the quintic

Quantum K-theory was developed by analogy with quantum cohomology (or Gromov–Witten theory), a theory that deforms the cohomology ring $H(X)$ of X and whose corresponding numerical invariants are rational numbers (known as Gromov–Witten invariants) or integers in the case of a Calabi–Yau threefold (known as the Gopakumar–Vafa invariants). A standard reference is [7] and the book [9]. For the quintic 3-fold X, the first six values of the GW and the GV invariants are given by

d	1	2	3	4	5	6
GW_d	$\frac{2875}{1}$	$\frac{4876875}{8}$	$\frac{8564575000}{27}$	$\frac{15517926796875}{64}$	$\frac{229305888887648}{1}$	$\frac{248249742157695375}{1}$
GV_d	2875	609250	317206375	242467530000	229305888887625	248249742118022000

with 2875 being the famous number of rational curves in the quintic. The two sets of invariants are related by the following multi-covering formula

$$\mathrm{GV}_n = \sum_{d|n} \frac{\mu(d)}{d^3}\, \mathrm{GW}_{n/d}, \qquad \mathrm{GW}_n = \sum_{d|n} \frac{1}{d^3}\, \mathrm{GV}_{n/d}.$$

In [38, Section 6.5], Tonita gave an algorithm to compute the quantum K-theory of the quintic and using it, he found that

$$\langle 1 \rangle_{0,1,1} = 2875,$$

where 2875 coincides with the famous number of lines in the quintic. Going further, (see Jockers–Mayr [29, 30] and equation (1.11) below) one finds that

$$\langle 1 \rangle_{0,1,2} = 620750 = 609250 + 4 \cdot 2875, \tag{1.5a}$$
$$\langle 1 \rangle_{0,1,3} = 317232250 = 317206375 + 9 \cdot 2875, \tag{1.5b}$$
$$\langle 1 \rangle_{0,1,4} = 242470013000 = 242467530000 + 4 \cdot 609250 + 16 \cdot 2875, \tag{1.5c}$$
$$\langle 1 \rangle_{0,1,5} = 229305888959500 = 229305888887625 + 25 \cdot 2875, \tag{1.5d}$$
$$\langle 1 \rangle_{0,1,6} = 248249743392434250$$
$$= 248249742118022000 + 4 \cdot 317206375 + 9 \cdot 609250 + 36 \cdot 2875 \tag{1.5e}$$

are nearly equal to GV invariants of the quintic, and more precisely matched with linear combinations of GV invariants. Surely this is not a coincidence and suggests that the GV invariants can fully reconstruct the quantum K-theory invariants. In [27] this "coincidence" is proven in abstractly. Givental and Tonita give a complete solution in genus-0 to the problem of expressing K-theoretic GW-invariants of a compact complex algebraic manifold in terms of its cohomological GW-invariants. One motivation for our work is to give an explicit formula (see Conjecture 1.1 below) of this abstract statement. To phrase our conjecture, recall that the rational K-theory of the quintic 3-fold X is given by

$$K(X) = \mathbb{Q}[x]/(x^4) \tag{1.6}$$

is the K-theory ring with basis $\{\Phi_\alpha\}$ for $\alpha = 0, 1, 2, 3$ where $\Phi_\alpha = x^\alpha$. Here $1 - x$ is the class of $\mathcal{O}(1)|_X$. We define

$$5a(d, r, q) = \frac{dr}{1 - q} + \frac{dq}{(1 - q)^2}, \tag{1.7a}$$

$$5b(d, r, q) = \frac{rd + r^2 - d}{1 - q} + \frac{d}{(1 - q)^2} - \frac{q + q^2}{(1 - q)^3}. \tag{1.7b}$$

Conjecture 1.1. *The small J-function of the quintic is expressed linearly in terms of the GV-invariants by*

$$\frac{1}{1-q} J(Q, q, 0) = 1 + x^2 \sum_{d,r \geq 1} a(d, r, q^r) \, \mathrm{GV}_d \, Q^{dr} + x^3 \sum_{d,r \geq 1} b(d, r, q^r) \, \mathrm{GV}_d \, Q^{dr}. \tag{1.8}$$

It is interesting to observe that the right hand side of (1.8) is a meromorphic function of q with poles at roots of unity of bounded order 3. In Section 3 we verify the above conjecture modulo $O(Q^7)$ by an explicit calculation. Without doubt, Conjecture 1.1 concerns not only the quintic 3-fold, but Calabi–Yau 3-folds with $h^{1,1} = 1$ (there are plenty of those, see, e.g., [2]) and beyond. In contrast to the case of $\mathbb{CP}^N$ (see (1.3)) or the case of Fano manifolds, the small J-function of the quintic is not hypergeometric. The above conjecture was formulated independently by Jockers–Mayr [29, p. 10] and a comparison between their formulation and ours is given in Section 3.3. Our conjecture also agrees with the results of Jockers–Mayr presented in [30, Table 6.1]. Let us introduce the following multi-covering notation

$$\mathrm{GV}_n^{(\gamma)} = \sum_{d|n} d^\gamma \, \mathrm{GV}_d \,.$$

Then, we have the following.

Corollary 1.2. *We have*

$$5J(Q, 0, 0) = 5 + x^2 \sum_{n=1}^{\infty} n \, \mathrm{GV}_n^{(0)} \, Q^n + x^3 \sum_{n=1}^{\infty} \left(n \, \mathrm{GV}_n^{(0)} + n^2 \, \mathrm{GV}_n^{(-2)} \right) Q^n$$

$$= 5 + \left(2875Q + 1224250Q^2 + 951627750Q^3 + 969872568500Q^4 + \cdots \right) x^2$$

$$+ \left(5750Q + 1845000Q^2 + 1268860000Q^3 + 1212342581500Q^4 + \cdots \right) x^3. \tag{1.9}$$

The above corollary reproduces the invariants of equations (1.5). To extract them, let $[J(Q, q, 0)]_{x^\alpha}$ denote the coefficient of x^α in $J(Q, q, 0)$. The next corollary is proven in Section 3.2.

Corollary 1.3. *We have*

$$\sum_{d \geq 1} \left\langle \frac{\Phi_\alpha}{1 - qL} \right\rangle_{0,1,d} Q^d = \begin{cases} -5[J(Q, q, 0)]_{x^2} + 5[J(Q, q, 0)]_{x^3} & \text{if} \quad \alpha = 0, \\ 5[J(Q, q, 0)]_{x^2} & \text{if} \quad \alpha = 1, \\ 0 & \text{if} \quad \alpha = 2, 3. \end{cases} \tag{1.10}$$

Setting $q = 0$, it follows that

$$\sum_{d \geq 1} \langle 1 \rangle_{0,1,d} Q^d = \sum_{n=1}^{\infty} n^2 \, \mathrm{GV}_n^{(-2)} \, Q^n = 2875Q + 620750Q^2 + 317232250Q^3 + 242470013000Q^4$$

$$+ 229305888959500Q^5 + 248249743392434250Q^6 + \cdots \tag{1.11}$$

matching with equations (1.5) (being the generating series of the K-theoretic versions of the GV-invariants, given in the second page and in [30, Table 6.1]), as well as

$$\sum_{d \geq 1} \langle \Phi_1 \rangle_{0,1,d} Q^d = \sum_{n=1}^{\infty} n \, \mathrm{GV}_n^{(0)} \, Q^n = 2875Q + 1224250Q^2 + 951627750Q^3 + 969872568500Q^4$$

$$+ 1146529444452500Q^5 + 1489498454615043000Q^6 + \cdots.$$

1.3 The linear q-difference equation for the quintic

In this section we give an explicit formula for the small linear q-difference equation for the quintic, assuming Conjecture 1.1. A key feature of this formula is that the coefficients of this equation are analytic (as opposed to polynomial) functions of Q and q. The small J-function $J(Q, q, 0)$, viewed as a vector in the vector space $K(X)$, forms the first column of the matrix $T(Q, q, 0)$ of fundamental solutions of the small linear q-difference equation in the basis $\{1, x, x^2, x^3\}$ of $K(X)$. The formula (1.8) for the small J-function and that fact that it is a cyclic vector of the linear q-difference equation allows us to reconstruct the matrix $A(Q, q, 0)$. See also [28, Theorem 1.1, Lemma 3.3]. To do so, let us introduce some useful notation. If $f = f(d, r, q) \in \mathbb{Q}(q)$ we denote

$$[f] = \sum_{d,r \geq 1} f(d, r, q^r)\, \mathrm{GV}_d\, Q^{dr}.$$

With this notation, equation (1.8) becomes

$$\frac{1}{1-q} J(Q, q, 0) = 1 + [a]x^2 + [b]x^3 = \begin{pmatrix} 1 \\ 0 \\ [a] \\ [b] \end{pmatrix}$$

in the basis $\{1, x, x^2, x^3\}$ of $K(X)$, where a, b are given by (1.7). Further, we denote $(Ef)(d, r, q) = q^d f(d, r, q)$, and define

$$5c = \pi_+((1 - E)a), \qquad 5d = \pi_+(Ea + (1 - E)b), \tag{1.12}$$

with projections $\pi_\pm \colon \mathcal{K}(q) \to \mathcal{K}_\pm(q)$ given in Section 2.1. Explicitly, we have

$$5c(d, r, q) = \frac{d^2}{1 - q},$$
$$5e(d, r, q) = \frac{dr}{1 - q} - \frac{d(dq + q - d)}{(1 - q)^2}.$$

Recall the T matrix from [28, Proposition 2.3] which is a fundamental solution of the linear q-difference equation, and whose first column is J. The proof of the next theorem and its corollary is given in Section 4.1.

Theorem 1.4. *Conjecture* 1.1 *implies that the small T-matrix of the quintic is given by*

$$T(Q, q, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ [a] & [c] & 1 & 0 \\ [b] & [e] & 0 & 1 \end{pmatrix} \tag{1.13}$$

and the small A-matrix of the linear q-difference equation is given by

$$A = I - D^{\mathrm{T}}, \qquad D(Q, q, 0) = \begin{pmatrix} 0 & 1 & [a - c - Ea] & [b - e + Ea - Eb] \\ 0 & 0 & 1 + [c - Ec] & [e + Ec - Ee] \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.14}$$

Note that the entries of $5D(Q, q, 0)$ are in $\mathbb{Z}[[Q]][q]$ and given explicitly in equations (4.2) below. Let us denote by $c_{ttt}(Q, q, t) = 5D_{2,3}(Q, q, t)$, where $D_{i,j}$ denotes the (i, j)-entry of the

matrix D. In other words, we have

$$c_{ttt}(Q,q) = 5 + \sum_{d,r \geq 1} d^2 \frac{1 - q^{dr}}{1 - q^r} \, \mathrm{GV}_d \, Q^{dr}$$

$$= \sum_{d=1}^{\infty} d^2 \, \mathrm{GV}_d \left(\mathrm{Li}_0 \left(Q^d \right) + \mathrm{Li}_0 \left(qQ^d \right) + \cdots + \mathrm{Li}_0 \left(q^{d-1}Q^d \right) \right),$$

where Li_s denotes the s-polylogarithm function $\mathrm{Li}_s(z) = \sum_{d \geq 1} z^d / d^s$. Recall the genus 0 generating series (minus its quadratic part) of the quintic [7, 9]

$$\mathcal{F}(Q) = \sum_{n=1}^{\infty} \mathrm{GW}_n \, Q^n = \frac{5}{6} (\log Q)^3 + \sum_{d=1}^{\infty} \mathrm{GV}_d \, \mathrm{Li}_3 \left(Q^d \right)$$

and its third derivative

$$c_{ttt}(Q) = (Q \partial_Q)^3 \mathcal{F}(Q) = 5 + \sum_{d=1}^{\infty} d^3 \, \mathrm{GV}_d \, \mathrm{Li}_0 \left(Q^d \right), \tag{1.15}$$

where $\partial_Q = \partial / \partial_Q$.

The next corollary gives a second relation between the $q = 1$ limit of quantum K-theory and quantum cohomology.

Corollary 1.5. *The function $c_{ttt}(Q, q) \in \mathbb{Z}[[Q]][q]$ is a q-deformation of the Yukawa coupling (i.e., 3-point function) $c_{ttt}(Q)$ in* (1.15). *Indeed, we have*

$$c_{ttt}(Q, 1) = c_{ttt}(Q), \qquad 5D_{2,3}(Q, q, 0) = c_{ttt}(Q, q).$$

Thus, the q-difference equation of the quantum K-theory of the quintic is a q-deformation of the well-known Picard–Fuchs equation of the quintic.

Let us abbreviate the four nontrivial entries of $D(Q, q, 0)$ by

$$\alpha = D_{1,3}, \qquad \beta = D_{1,4}, \qquad \gamma = D_{2,3}, \qquad \delta = D_{2,4}.$$

Lemma 1.6 ([30, equations (8.22) and (8.23)]). *The linear q-difference equation*

$$\Delta \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \alpha & \beta \\ 0 & 0 & \gamma & \delta \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

(where $\Delta = 1 - E$) is equivalent to the equation

$$\mathcal{L}y_0 = 0, \qquad \mathcal{L} = \Delta \left(1 + \Delta \frac{\delta + E\alpha + \Delta\beta}{\gamma + \Delta\alpha} \right)^{-1} \Delta(\gamma + \Delta\alpha)^{-1}\Delta^2. \tag{1.16}$$

We now discuss the $q \to 1$ limit, using the realization of the q-commuting operators $E = \mathrm{e}^{hQ\partial_Q}$ and Q which act on a function $f(z, h)$ by

$$(Ef)(z, h) = f(z + h, h), \qquad (Qf)(z, h) = \mathrm{e}^z f(z, h), \qquad EQ = \mathrm{e}^h QE,$$

where $Q = \mathrm{e}^z$ and $q = \mathrm{e}^h$. Then, in the limit $h \to 0$, the operator $\mathcal{L}$ is given by

$$\mathcal{L}(\Delta, Q, q) = \frac{1}{\gamma(Q, 1)} \Delta^4 + \partial_z^2 \frac{1}{\gamma(Q, 1)} \partial_z^2 h^4 + O\left(h^5\right), \tag{1.17}$$

where $5\gamma(Q, 1) = c_{ttt}(Q, 1)$. Thus, the coefficient of h^4 is the Picard–Fuchs equation of the quintic, whereas the coefficient of h^0 (the analogue of the AJ conjecture) is a line $(1 - E)^4 = 0$ with multiplicity 4, punctured at the zeros of $\gamma(Q, 1) = 0$. It is not clear if one can apply topological recursion on such a degenerate curve.

2   A review of quantum K-theory

2.1   Notation

In this section we collect some useful notation that we use throughout the paper. For a smooth projective variety X, let $K(X) = K^0(X; \mathbb{Q})$ denote the Grothendieck group of topological complex vector bundles with rational coefficients.

Although we will not use it, the Chern class map induces a rational isomorphism of rings

$$\mathrm{ch}\colon\ K(X) \otimes \mathbb{Q} \to H^{\mathrm{ev}}(X, \mathbb{Q})$$

between K-theory and even cohomology. The ring $K(X)$ has a basis $\{\Phi_\alpha\}$ for $\alpha = 0, \ldots, N$ such that $\Phi_0 = 1 = [\mathcal{O}_X]$ is the identity element. There is a nondegenerate pairing on $K(X)$ given by $(E, F) \in K(X) \otimes K(X) \mapsto \chi(E \otimes F)$, where

$$\chi(E) = \int_X \mathrm{ch}(E)\mathrm{td}(X)$$

is the holomorphic Euler characteristic of E. Let $\{\Phi^\alpha\}$ denote the dual basis of $K(X)$ with respect to the above pairing. Let $\{P_1, \ldots, P_r\}$ denote a collection of vector bundles whose first Chern class forms a nef integral basis of $H^2(X, \mathbb{Z})/\text{torsion}$, and let $Q = (Q_1, \ldots, Q_r)$ be the collection of Novikov variables dual to $(P_1, \ldots, P_r)$.

The vector space $\mathcal{K}(q) = \mathbb{Q}(q)$ admits a symplectic form

$$\omega(f, g) = (\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty})\left(f(q)g(q^{-1})\frac{\mathrm{d}q}{q}\right)$$

and a splitting

$$\mathcal{K}(q) = \mathcal{K}_+(q) \oplus \mathcal{K}_-(q)$$

(with projections $\pi_\pm\colon \mathcal{K}(q) \to \mathcal{K}_\pm(q)$) into a direct sum of two Lagrangian susbpaces $\mathcal{K}_+(q) = \mathbb{Q}[q^{\pm 1}]$ and $\mathcal{K}_-(q)$, the space of reduced functions of q, i.e., rational functions of negative degree which are regular at $q = 0$.

2.2   Reconstruction theorems for quantum K-theory

In our paper we will focus exclusively on the genus 0 quantum K-theory of X (i.e., $g = 0$ in (1.1)). The collection of integers (1.1) can be encoded in several generating series. Among them is the primary potential

$$\mathcal{F}_X(Q, t) = \sum_{d,n}\langle t, \ldots, t\rangle_{0,n,d}\frac{Q^d}{n!} \in \mathbb{Q}[[Q, t]]$$

(where the summation is over $d \in \mathrm{Eff}(X)$ and $n \geq 0$), the J-function

$$J_X(Q, q, t) = (1 - q)\Phi_0 + t + \sum_{d,n}\sum_{\alpha}\left\langle t, \ldots, t, \frac{\Phi_\alpha}{1 - qL}\right\rangle_{0,n+1,d}\Phi^\alpha Q^d \in K(X) \otimes \mathcal{K}(q)[[Q, t]]$$

(where $\{\Phi_\alpha\}$ is a basis for $K(X)$ for $\alpha = 0, \ldots, N$ with $\Phi_0 = 1$), and the T matrix $T_{\alpha,\beta}(Q, q, t) \in \mathrm{End}(K(X)) \otimes \mathcal{K}(q)[[Q, t]]$ and its inverse, whose definition we omit but may be found in [28, Section 2]. We may think of $\mathcal{F}_X$, $J_X(Q, q, t)$ and $T(Q, q, t)$ as scalar-valued, vector-valued and matrix-valued invariants, respectively. $J_X(Q, q, t)$ specializes to $J_X(Q, q, 0)$ when $t = 0$ and

specializes to $\mathcal{F}_X(Q, t)$ when $\alpha = 0$ (as follows from the string equation). Also, the $\alpha = 0$ column of T is J_X.

There are several reconstruction theorems that determine all the invariants (1.1) from others. In [28, Theorem 1.1], it was shown that the small J-function $J_X(Q, q, 0)$ uniquely determines the J-function $J_X(Q, q, t)$, the primary potential $\mathcal{F}_X(Q, t)$ and the integers (1.1) (with $g = 0$), under the assumption that $K(X)$ is generated by line bundles. In [24] it was shown (under the same assumption on X) that the small J-function $J_X(Q, q, 0)$ reconstructs a permutation-equivariant version of the quantum K-theory of X. This theory was introduced by Givental in [24], where this theory takes into account the action of the symmetric groups S_n on the moduli spaces $\overline{\mathcal{M}}_{g,n}^{X,d}$ that permutes the marked points. The J function of the permutation-equivariant quantum K-theory of X takes values in the ring $K(X) \otimes \mathcal{K}(q) \otimes \Lambda[[Q]]$ where Λ is the ring of symmetric functions in infinitely many variables [34]. $K(X)$, $\mathbb{Q}[[Q]]$ and Λ are λ-rings with Adams operations ψ^r, so is their tensor product. Moreover, the small J function of the permutation-equivariant quantum K-theory of X agrees with the small J-function $J_X(Q, q, 0)$ of the (ordinary) genus 0 quantum K-theory of X. According to a reconstruction theorem of Givental [24] one can recover all genus zero permutation-equivariant K-theoretic GW invariants of a projective manifold X (under the mild assumption that the ring $K(X)$ is generated by line bundles) from any point t^* on their K-theoretic Lagrangian cone via an explicit flow. In fortunate situations (that apply to the quintic as we shall see below), one is given a value $J_X(Q, q, t^*) \in K(X) \otimes \mathcal{K}(q)[[Q]] \subset K(X) \otimes \mathcal{K}(q) \otimes \Lambda[[Q]]$ and $t^* \in K(X) \otimes \mathcal{K}_+(q)[[Q]]$ (e.g., $t^* = 0$), in which case there exists a unique $\varepsilon(x, Q, q) \in K(X) \otimes Q\mathcal{K}_+(q)[[Q]]$ such that for all t

$$J_X(Q, q, t) = \exp\left(\sum_{r \geq 1} \frac{\psi^r(\varepsilon((1-x)E, Q, q))}{r(1-q^r)}\right) J_X(Q, q, t^*) \in K(X) \otimes \mathcal{K}(q)[[Q]], \qquad (2.1)$$

where E is the operator that shifts Q to qQ. The key point here is that the coefficients of $\varepsilon(x, Q, q)$ (for each power of Q and x) are in the subspace $\mathcal{K}_+(q)$ of $\mathcal{K}(q)$ whereas the corresponding coefficients of $J_X(Q, q, t)$ are in the complementary subspace $\mathcal{K}_-(q)$ of $\mathcal{K}(q)$. Another key point is that although the above formula a priori is an equality in the permutation-equivariant quantum K-theory, in fact it is an equality of the ordinary quantum K-theory when ε is independent of Λ.

It follows that a single value $J_X(Q, q, t^*) \in K(X) \otimes \mathcal{K}(q)[[Q]]$ uniquely determines t^* as well as the small J-function $J_X(Q, q, 0)$, which in turn determines the permutation-equivariant J-function $J_X(Q, q, t)$ for all t via (2.1).

2.3   A special value for the J-function of the quintic

For concreteness, we will concentrate on the case X of the quintic. To use the above formula (2.1) we need the value $J_X(Q, q, t^*)$ at some point t^*. Such a value was given by Givental in [23, p. 11] and by Tonita in [38, Theorem 1.3 and Corollary 6.8] who proved that if J_d denotes the coefficient of Q^d in $J_{\mathbb{CP}^4}(Q, q, 0)$ given in (1.3), then

$$I_{\mathcal{O}(5)}(Q, q) = \sum_{d=0}^{\infty} J_d\left((1-x)^5 q; q\right)_{5d} Q^d = (1-q) \sum_{d=0}^{\infty} \frac{\left((1-x)^5 q; q\right)_{5d}}{\left((1-x)q; q\right)_d^5} Q^d \qquad (2.2)$$

lies on the K-theoretic Lagrangian cone of the quintic X. This means that if $\iota \colon X \to \mathbb{CP}^4$ is the inclusion, and $\iota^* \colon K(\mathbb{CP}^4) = \mathbb{Q}[x]/(x^5) \to K(X) = \mathbb{Q}[x]/(x^4)$ is the induced map (sending $x \bmod x^5$ to $x \bmod x^4$), there exists a t^* such that $\iota^* I_{\mathcal{O}(5)}(Q, q) = J_X(Q, q, t^*)$. In other words, we have

$$J(Q, q, t^*) = (1-q) \sum_{d=0}^{\infty} \frac{\left((1-x)q; q\right)_{5d}}{\left((1-x)q; q\right)_d^5} Q^d \in K(X) \otimes \mathcal{K}(q)[[Q]]. \qquad (2.3)$$

Interestingly, the above formula has been interpreted by Jockers and Mayr as an example of the 3d-3d correspondence of gauged linear σ-models [30]. More precisely, the disk partition function of a 3d gauged linear σ-model is a one-dimensional (so-called vortex) integral whose integrand is a ratio of infinite Pochhammer symbols. A residue calculation then produces the q-hypergeometric series (2.2).

3 The flow of the J-function

3.1 Implementing the flow

In this section we explain how to obtain a formula for the small J-function of the quintic (one power of Q at a time) using formula (2.2) and the flow (2.1). Observe that the coefficients of q in the function $J(Q, q, t^*)$ given in (2.3) are not in $\mathcal{K}_-(q)$. For instance,

$$\mathrm{coeff}\left(\frac{1}{1-q}J(Q,q,t^*), x^0\right) = \sum_{d=0}^{\infty} \frac{(q;q)_{5d}}{(q;q)_d^5} Q^d$$

is a power series in Q whose coefficients are in $\mathcal{K}_+(q)$ (and even in $\mathbb{N}[q]$) and not in $\mathcal{K}_-(q)$. Note also that the function $J(Q, q, t^*)$ satisfies a 24th order (but *not* a 4th order) linear q-difference equation with polynomial coefficients. This is discussed in detail in Section 4.2 below.

To find $J(Q, q, 0)$ from $J(Q, q, t^*)$, we need to apply a flow operator (2.1). To state the theorem, recall that $K(X) \otimes \mathcal{K}(q)[[Q]]$ is a λ-ring with Adams operations $\psi^{(r)}$ given by combining the usual Adams operations in K-theory with the replacement of Q and q by Q^r and q^r. More precisely, for a positive natural number r, we have

$$\psi^{(r)}\colon \ K(X) \otimes \mathcal{K}(q)[[Q]] \to K(X) \otimes \mathcal{K}(q)[[Q]], \psi^{(r)}\big((1-x)^i f(q)Q^j\big) = (1-x)^{ri} f(q^r)Q^{rj}$$

for $f(q) \in \mathcal{K}(q)$ and natural numbers i, j and x as in (1.6). Recall that the plethystic exponential of $f(x, Q, q) \in K(X) \otimes \mathcal{K}(q)[[Q]]$ (with $f(x, 0, q) = 0$) is given by

$$\mathrm{Exp}(f) = \exp\left(\sum_{r=1}^{\infty} \frac{\psi^{(r)}(f)}{r}\right).$$

It is easy to see that when f is small (i.e., $f(x, 0, q) = 0$), then $\mathrm{Exp}(f) \in K(X) \otimes \mathcal{K}(q)[[Q]]$ is well-defined. Let E denote the q-difference operator that shifts Q to qQ, as in (1.12). By slight abuse of notation, we denote

$$\begin{aligned} E\colon \ &K(X) \otimes \mathcal{K}(q)[[Q]] \to K(X) \otimes \mathcal{K}(q)[[Q]], \\ &E\big((1-x)^i f(Q)Q^j\big) = (1-x)^i f(qQ)Q^j. \end{aligned} \tag{3.1}$$

Throughout the paper, the operators E and Q will act on a function $f(Q, q)$ by

$$(Ef)(Q,q) = f(qQ, q), \qquad (Qf)(Q,q) = Qf(Q,q), \qquad EQ = qQE. \tag{3.2}$$

The theorem of Givental–Tonita asserts that there exists a unique

$$\varepsilon(x, Q, q) \in K(X) \otimes Q\mathcal{K}_+(q)[[Q]]$$

such that

$$\mathrm{Exp}\left(\frac{\varepsilon((1-x)E, Q, q)}{1-q}\right) J(Q, q, t^*) \in K(X) \otimes \mathcal{K}_-(q)[[Q]] \tag{3.3}$$

and then, the left hand side of the above equation is $J(Q, q, 0)$. Equation (3.3) is a non-linear fixed-point equation for ε that has a unique solution that may be found working on one Q-degree at a time. Indeed, we can write

$$\varepsilon(x, Q, q) = \sum_{k=1}^{\infty} \varepsilon_k(x, q) Q^k, \qquad \varepsilon_k(x, q) = \sum_{\ell=0}^{3} \sum_{k=1}^{\infty} \varepsilon_{k,\ell}(q) x^\ell Q^k.$$

Then for each positive integer number N we have

$$\pi_+ \left(\exp \left(\sum_{r=1}^{N} \sum_{\ell=0}^{3} \sum_{k=1}^{N} \frac{\psi^{(r)} \varepsilon_{k,\ell}(q)}{r(1-q^r)} Q^{rk} ((1-x)E)^{\ell r} \right) J(Q, q, t^*) \right) = 0.$$

Equating the coefficient of each power of x^i for $i = 0, \ldots, 3$ to zero in the above equation, we get a system of four inhomogeneous linear equations with unknowns $(\varepsilon_{N,0}, \ldots, \varepsilon_{N,3})$ (with coefficients polynomials in $\varepsilon_{N',\ell'}$ for $N' < N$), with a unique solution in the field $\mathcal{K}(q)$. A further check (according to Givental–Tonita's theorem) is that the unique solution lies in $\mathcal{K}_+(q)$, and even more, in our case we check that it lies in $\mathbb{Q}[q]$. Once $\varepsilon_{N'}(x, q)$ is known for $N' \leq N$, equation (3.3) allows us to compute $J_d(q)$, where

$$J(Q, q, 0) = \sum_{d=0}^{\infty} J_d(q) Q^d.$$

For instance, when $N = 1$ we have

$$
\begin{aligned}
\varepsilon_{1,0}(q) &= 1724 + 572q - 625q^2 - 1941q^3 - 3430q^4 - 4952q^5 - 6223q^6 - 6755q^7 - 6184q^8 \\
&\quad - 4690q^9 - 2747q^{10} - 969q^{11}, \\
\varepsilon_{1,1}(q) &= -4600 - 1140q + 2485q^2 + 6520q^3 + 11140q^4 + 15890q^5 + 19860q^6 + 21490q^7 \\
&\quad + 19630q^8 + 14860q^9 + 8690q^{10} + 3060q^{11}, \\
\varepsilon_{1,2}(q) &= 4025 + 555q - 3115q^2 - 7255q^3 - 12055q^4 - 17020q^5 - 21175q^6 - 22850q^7 \\
&\quad - 20830q^8 - 15740q^9 - 9190q^{10} - 3230q^{11}, \\
\varepsilon_{1,3}(q) &= -1150 + 10q + 1250q^2 + 2670q^3 + 4340q^4 + 6080q^5 + 7540q^6 + 8120q^7 \\
&\quad + 7390q^8 + 5575q^9 + 3250q^{10} + 1140q^{11},
\end{aligned}
$$

and, consequently, we find that

$$
\begin{aligned}
J_0(q) &= 1 - q, \\
J_1(q) &= -\frac{575x^2}{-1+q} - \frac{1150(-1+2q)x^3}{(-1+q)^2}
\end{aligned}
$$

in agreement with [30, eqation (6.38)]. Continuing our computation, we find that

$$
\begin{aligned}
J_2(q) = &-\frac{25\left(9794 + 19496q + 9725q^2\right)x^2}{(-1+q)(1+q)^2} \\
&- \frac{50\left(-7380 - 9748q + 14760q^2 + 29244q^3 + 12139q^4\right)x^3}{(-1+q)^2(1+q)^3}
\end{aligned}
$$

and

$$J_3(q) = -\frac{25\left(7613022 + 15225906q + 22838859q^2 + 15225860q^3 + 7612953q^4\right)x^2}{(-1+q)\left(1+q+q^2\right)^2}$$

$$- \frac{50}{(-1+q)^2 \left(1+q+q^2\right)^3} \left(-5075440 - 7612953q - 7612953q^2 + 10150880q^3\right.$$
$$\left. + 22838859q^4 + 30451812q^5 + 17763442q^6 + 7612953q^7\right)x^3.$$

Two further values of $J_d(q)$ for $d = 4, 5$ were computed but are too long to be presented here. Based on this data, we guessed the formula for $J(Q, q, 0)$ given in (1.8). Finally, we computed $J_6(q)$ and found that it is in agreement with out predicted formula (1.8).

3.2   Extracting quantum K-theory counts from the small J-function

In this section we give a proof of Corollary 1.3 for the quintic X. Recall that $K(X)$ from equation (1.6) has basis $\Phi_\alpha = x^\alpha$ for $\alpha = 0, 1, 2, 3$ with $x^4 = 0$ and inner product

$$(\Phi_a, \Phi_b) = \int_X \Phi_a \Phi_b \mathrm{td}(X) = \begin{pmatrix} 0 & 5 & -5 & 5 \\ 5 & -5 & 5 & 0 \\ -5 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

The dual basis $\{\Phi^a\}$ of $K(X)$ is given by

$$\Phi^0 = \tfrac{1}{5}\Phi_3, \qquad \Phi^1 = \tfrac{1}{5}(\Phi_2 + \Phi_3), \qquad \Phi^2 = \tfrac{1}{5}(\Phi_1 + \Phi_2), \qquad \Phi^3 = \tfrac{1}{5}(\Phi_0 + \Phi_1 - \Phi_3), \tag{3.5}$$

and is related to the basis $\{\Phi_a\}$ by

$$\Phi_0 = 5\left(\Phi^1 - \Phi^2 + \Phi^3\right), \qquad \Phi_1 = 5\left(\Phi^0 - \Phi^1 + \Phi^2\right),$$
$$\Phi_2 = 5\left(-\Phi^0 + \Phi^1\right), \qquad \Phi_3 = 5\Phi^0.$$

Substituting Φ^α as above in equation (1.2) and collecting the powers of x^α, it follows that

$$[J(Q, q, 0)]_1 = 1 - q + \frac{1}{5} \sum_{d \geq 1} \left\langle \frac{\Phi_3}{1 - qL} \right\rangle_{0,1,d} Q^d,$$

$$[J(Q, q, 0)]_x = \frac{1}{5} \sum_{d \geq 1} \left(\left\langle \frac{\Phi_2}{1 - qL} \right\rangle_{0,1,d} + \left\langle \frac{\Phi_3}{1 - qL} \right\rangle_{0,1,d} \right) Q^d,$$

$$[J(Q, q, 0)]_{x^2} = \frac{1}{5} \sum_{d \geq 1} \left(\left\langle \frac{\Phi_1}{1 - qL} \right\rangle_{0,1,d} + \left\langle \frac{\Phi_2}{1 - qL} \right\rangle_{0,1,d} \right) Q^d,$$

$$[J(Q, q, 0)]_{x^3} = \frac{1}{5} \sum_{d \geq 1} \left(\left\langle \frac{\Phi_0}{1 - qL} \right\rangle_{0,1,d} + \left\langle \frac{\Phi_1}{1 - qL} \right\rangle_{0,1,d} - \left\langle \frac{\Phi_3}{1 - qL} \right\rangle_{0,1,d} \right) Q^d.$$

The above is a linear system of equations with unknowns $\sum_{d \geq 1} \left\langle \frac{\Phi_\alpha}{1-qL} \right\rangle_{0,1,d} Q^d$ for $\alpha = 0, 1, 2, 3$. Solving the linear system combined with equation (1.9), gives (1.10). Setting $q = 0$ in (1.10) and using Corollary 1.2, we obtain (1.11) and (1.3) and conclude the proof of Corollary 1.3.

3.3   A comparison with Jockers–Mayr

In this section we give the details of the comparison of our Conjecture 1.1 with a conjecture of Jockers–Mayr [29, p. 10].

To begin with, their $I_{QK}(t)$ is our $J(Q, q, t)$ and their $I(0)$ in [29, equation (7)] is our $J(Q, q, 0)$. They drop the index QK later on. From [29, equation (4)] it follows that they are working in the same basis $\Phi_\alpha = x^a$, $\alpha = 0, 1, 2, 3$, as we are. Furthermore, the inner product on $K(X)$

[29, equation (6)] agrees with the one given in equation (3.4) with dual basis $\{\Phi^a\}$ of $K(X)$ given in (3.5). By [29, equation (8)], specialized to the quintic, the function $I(t)$ becomes $I(t) = 1 - q + t\Phi_1 + F^2(t)\Phi_2 + F^3(t)\Phi_3$. Then, they define functions F_A and $\hat{F}^A$ by writing $\sum_A F^A \Phi_A = \sum_A (F_{A,cl} + \hat{F}_A)\Phi^A$, where $\hat{F}_A(t) = \sum_{d>0} Q^d \langle\!\langle \frac{\Phi_A}{1-qL} \rangle\!\rangle_d$, cf. [29, equation (9)], and $F_{A,cl}$ are "constant", i.e., independent of Q and t. Note that only F^2, F^3 are nonzero which implies that only F_0, F_1 are nonzero. Their conjecture [29, p. 10] can now be stated (in the case of the quintic) as follows [29, equation (10)]:

$$\hat{F}_0 = p_2 + \frac{1}{(1-q)^2}[(1-3q)\mathcal{F} + qt\mathcal{F}_1]_{t^n>2},$$

$$\hat{F}_1 = p_{1,1} + \frac{1}{(1-q)}[\mathcal{F}_1]_{t^n>1},$$

where p_2, $p_{1,1}$, $\mathcal{F}$, $\mathcal{F}_1$ are certain explicitly given functions of t and the Gopakumar–Vafa invariants GV_d, [29, equations (11) and (12)]. Combining everything so far, their conjecture reads

$$I(t) = 1 - q + t\Phi_1 + (F_{1,cl} + p_{1,1} + \frac{1}{(1-q)}[\mathcal{F}_1]_{t^n>1})\Phi^1$$

$$+ (F_{0,cl} + p_2 + \frac{1}{(1-q)^2}[(1-3q)\mathcal{F} + qt\mathcal{F}_1]_{t^n>2})\Phi^0.$$

We will not spell out these functions completely, but only their value at $t = 0$ in order to compare it to our formulas. First, the brackets $[\ldots]_{t^n>1}$, $[\ldots]_{t^n>2}$ vanish for $t = 0$. So we are left with p_2 and $p_{1,1}$ [29, equation (12)]. Noting that $\sum_j d_j t_j = 0$ for $t = 0$, these read

$$\frac{1}{1-q}p_{1,1}|_{t=0} = \sum_{d>0} Q^d \sum_{r|d} \mathrm{GV}_{d/r} \frac{d(1-q^r) + \frac{d}{r}q^r}{(1-q^r)^2},$$

$$\frac{1}{1-q}p_2|_{t=0} = \sum_{d>0} Q^d \sum_{r|d} \mathrm{GV}_{d/r} \frac{r^2(1-q^r)^2 - q^r(1+q^r)}{(1-q^r)^3}.$$

Next, we rewrite these sums so that they run over all values of r

$$\frac{1}{1-q}p_{1,1}|_{t=0} = \sum_{d,r>0} Q^{dr}\, \mathrm{GV}_r \frac{dr(1-q^r) + dq^r}{(1-q^r)^2},$$

$$\frac{1}{1-q}p_2|_{t=0} = \sum_{d,r>0} Q^{dr}\, \mathrm{GV}_r \frac{r^2(1-q^r)^2 - q^r(1+q^r)}{(1-q^r)^3}.$$

Hence,

$$\frac{1}{1-q}p_{1,1}|_{t=0} = 5 \sum_{d,r>0} Q^{dr}\, \mathrm{GV}_r\, a(d,r,q^r),$$

$$\frac{1}{1-q}p_2|_{t=0} = 5 \sum_{d,r>0} Q^{dr}\, \mathrm{GV}_r\, (b(d,r,q^r) - a(d,r,q^r)).$$

The appearance of the term involving $a(d,r,q^r)$ in the second equation is due to the change of basis $\Phi_2 = 5(-\Phi^0 + \Phi^1)$. This completes the compatibility of our conjecture and theirs.

4 q-difference equations

4.1   The small q-difference equation of the quintic

In this section we explain how Theorem 1.4 follows from Conjecture 1.1. We begin with a general discussion. Given a collection of vector functions $f_j(Q,q) \in \mathbb{Q}(q)[[Q]]^r$ for $j = 1,\ldots,r$ such that

$\det(f_1|f_2|\dots|f_r)$ is not identically zero, there is always a canonical linear q-difference equation

$$(Ey)(Q,q) = A(Q,q)y(Q,q)$$

with fundamental solution set $f_1,\dots,f_r$, where E is the shift operator of equation (3.1) that replaces Q by qQ. Indeed, the equations $Ey_j = Ay_j$ for $j = 1,\dots,r$ are equivalent to the matrix equation $ET = AT$ where $T = (f_1|f_2|\dots|f_r)$ is the fundamental matrix solution, and inverting T, we find that $A = (ET)^{-1}T$. This can be applied in particular to the case of a collection $E^j g$ for $j = 0,\dots,r-1$ of a vector function $g(Q,q) \in \mathbb{Q}(q)[[Q]]$ that satisfies $\det\big(g|Eg|\dots|E^{r-1}g\big)$ is nonzero. Said differently, every vector function $g(Q,q) \in \mathbb{Q}(q)[[Q]]$ along with its $r-1$ shifts (generically) satisfies a linear q-difference equation.

We will apply the above principle to the 4-tuple $((1-x)E)^j J(Q,q,0)/(1-q) \in K(X) \otimes \mathcal{K}(q)[[Q]]$ for $j = 0,\dots,3$ where $J(Q,q,0) \in K(X) \otimes \mathcal{K}_-(q)[[Q]]$ is as in Conjecture 1.1. However, notice that although the q-coefficients of $J(Q,q,0)/(1-q)$ are in $\mathcal{K}_-(q)$, this is no longer true for the shifted functions $((1-x)E)^j J(Q,q,0)/(1-q)$ for $j = 1,2,3$. In that case, we need to apply the Birkhoff factorization [25, App.A] to the matrix

$$\frac{1}{1-q}\big(J|(1-x)EJ|((1-x)E)^2J|((1-x)E)^3J\big) = TU, \tag{4.1}$$

where the q-coefficients of the entries of T are in $\mathcal{K}_-(q)$ and of U are in $\mathcal{K}_+(q)$ (compare also with Lemma 3.3 of [28, equation (4)]). The existence and uniqueness of matrices T and U in the above equation follows from the fact that the left hand side of the above equation is unipotent, and the proof is discussed in detail in the above reference.

In our case, the choice

$$T = \frac{1}{1-q}\pi_+\big(J|(1-x)EJ|((1-x)E)^2J|((1-x)E)^3J\big)$$

together with equation (4.1) implies that the q-coefficients of the entries of U are in $\mathcal{K}_+(q)$. Equation (1.13) for the fundamental matrix T follows from the fact that

$$\pi_+\left(q^d\left(\frac{r^2}{1-q} - \frac{q+q^2}{(1-q)^3}\right)\right) = \frac{-1+3q-4q^2}{(1-q)^3} + \frac{(-1+d)(-1-d+3q+dq)}{(1-q)^2} + \frac{r^2}{1-q},$$

$$\pi_+\left(q^d\left(\frac{r}{1-q} + \frac{q}{(1-q)^2}\right)\right) = \frac{-d+q+dq}{(1-q)^2} + \frac{r}{1-q}$$

valid for all positive natural numbers d and r.

Having computed the fundamental matrix T (1.13), we use [28, equation (2)], with $P^{-1}q^{Q\partial_Q}$ replaced by $1-(1-x)E$ to deduce the small A-matrix (1.14).

Explicitly, the four nontrivial entries of the matrix D are given by

$$5(a-c-Ea)(d,r,q) = \frac{d\big(-d+q+dq-q^{1+d}+r-qr-q^dr+q^{1+d}r\big)}{(1-q)^2}, \tag{4.2a}$$

$$5(b-e+Ea-Eb)(d,r,q) = -\frac{q^2\big(1+2d+d^2-r^2\big)+q\big(1-2d-2d^2+2r^2\big)}{(1-q)^3}$$

$$+ \frac{d^2-r^2+q^d\big(-q-q^2+r^2-2qr^2+q^2r^2\big)}{(1-q)^3}, \tag{4.2b}$$

$$5(c-Ec)(d,r,q) = \frac{d^2\big(1-q^d\big)}{1-q}, \tag{4.2c}$$

$$5(e+Ec-Ee)(d,r,q) = -\frac{d\big(-d+q+dq-q^{1+d}-r+qr+q^dr-q^{1+d}r\big)}{(1-q)^2}. \tag{4.2d}$$

Note that the entries of $5D$ are in $\mathbb{Z}[[Q]][q]$. Moreover, the values when $q = 1$ are given by

$$5(a - c - Ea)(d, r, 1) = -\frac{1}{2}d^2(1 + d - 2r),$$

$$5(b - e + Ea - Eb)(d, r, 1) = -\frac{1}{6}d(1 + 3d + 2d^2 - 6r^2),$$

$$5(c - Ec)(d, r, q) = d^3,$$

$$5(e + Ec - Ee)(d, r, 1) = \frac{1}{2}d^2(1 + d + 2r).$$

As a further consistency check, note that our matrix D given in (1.14) equals to the matrix D of [30, equation (8.21)].

Given the formula of (1.14), an explicit calculation shows that the entries of D are given by (4.2). This concludes the proof of Theorem 1.4.

Proof of Corollary 1.5. It follows from equations (4.2c) and (1.15).   ∎

Proof of Lemma 1.6. We have

$$\Delta y_0 = y_1 + \alpha y_2 + \beta y_3, \qquad \Delta y_1 = \gamma y_2 + \delta y_3, \qquad \Delta y_2 = y_3, \qquad \Delta y_3 = 0.$$

The lemma follows by eliminating y_1, y_2, y_3 (one at a time) using the fact that

$$E(fg) = (Ef)(Eg), \qquad \Delta(fg) = (\Delta f)g + f(\Delta g) - (\Delta f)(\Delta g).$$

(which follows from $(Ef)(Q, q) = f(qQ, q)$ and $\Delta = 1 - E$). Indeed, we have

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 + \alpha y_2 + \beta y_3) = (\gamma + \Delta\alpha)y_2 + (\delta + E\alpha + \Delta\beta)y_3,$$

and hence,

$$(\gamma + \Delta\alpha)^{-1}\Delta^2 y_0 = y_2 + \frac{\delta + E\alpha + \Delta\beta}{\gamma + \Delta\alpha}y_3,$$

and hence,

$$\Delta(\gamma + \Delta\alpha)^{-1}\Delta^2 y_0 = \left(1 + \frac{\delta + E\alpha + \Delta\beta}{\gamma + \Delta\alpha}\right)y_3.$$

Applying Δ once again and using $\Delta y_3 = 0$ concludes the proof of equation (1.16). Note that the notation is such that an operator Δ is applied to everything on the right hand side.

The $q = 1$ limit of $\mathcal{L}(\Delta, Q, q)$ follows from equation (1.16), the fact that

$$(\Delta f)(Q, q)|_{q=1} = (f(qQ, q) - f(Q, q))|_{q=1} = 0$$

and Corollary 1.5.   ∎

4.2 The Frobenius method for linear q-difference equations

In this section we discuss in detail the linear q-difference equation satisfied by the function $J(Q, q, t^*)$ of (2.2). Recall the operators E and Q that act on functions of Q and q by (3.2). Let

$$J(Q, q, x) = \sum_{n=0}^{\infty} a_n(q, x)Q^n = J_0(Q, q) + J_1(Q, q)x + \cdots \in \mathbb{Q}(q)[[Q, x]], \tag{4.3}$$

where $J_n(Q,q) \in \mathbb{Q}(q)[[Q]]$ for all n and

$$a_n(q,x) = \frac{\left(e^{5x}q; q\right)_{5n}}{\left(e^x q; q\right)_n^5},$$

where e^{ax} is to be understood as a polynomial in x obtained as $e^{ax} + O(x^4)$.

The functions $J_n(Q,q)$ are given by series whose summand ia a q-hypergeometric function times a polynomial of q-harmonic functions. For example, we have

$$J_0(Q,q) = \sum_{n=0}^{\infty} \frac{(q;q)_{5n}}{(q;q)_n^5} Q^n,$$

$$J_1(Q,q) = \sum_{n=0}^{\infty} \frac{(q;q)_{5n}}{(q;q)_n^5}(1 + 5H_{5n}(q) - 5H_n(q))Q^n,$$

where $H_n(q) = \sum_{j=1}^{n} q^j/\left(1-q^j\right)$ is the nth q-harmonic number. Consider the 25-th order linear q-difference operator

$$L_5(E, Q, q) = (1 - E)^5 - Q \prod_{j=1}^{5} \left(1 - q^j E^5\right) \tag{4.4}$$

with coefficients polynomials in Q and q. Note that $L_5 = (1 - E)^5 - \prod_{j=1}^{5}\left(1 - q^{5-j}E^5\right)Q$, hence L_5 factors as $1 - E$ times a 24-th order operator.

Lemma 4.1. *With J as in* (4.3) *and L_5 as in* (4.4), *we have*

$$L_5\left(e^x, Q, q\right)J = \left(1 - e^x\right)^5.$$

Proof. It is easy to see that

$$\frac{a_n(q,x)}{a_{n-1}(q,x)} = \frac{\prod_{j=1}^{5}\left(1 - e^{5x}q^{5n-j}\right)}{\left(1 - e^x q^n\right)^5}.$$

Hence,

$$\left(1 - e^x q^n\right)^5 a_n(q,x)Q^n = Q\prod_{j=1}^{5}\left(1 - e^{5x}q^{5n-j}\right)a_{n-1}(q,x)Q^{n-1}$$

and in operator form,

$$\left(1 - e^x E\right)^5 a_n(q,x)Q^n = Q\prod_{j=1}^{5}\left(1 - q^j e^{5x}E^5\right)a_{n-1}(q,x)Q^{n-1}.$$

Summing from $n = 1$ to infinity, we obtain that

$$\left(1 - e^x E\right)^5 (J - 1) = Q\prod_{j=1}^{5}\left(1 - q^j e^{5x}E^5\right)J.$$

Since $\left(1 - e^x E\right)^5 = \left(1 - e^x\right)^5$, the result follows. ∎

Note that the proof of Lemma 4.1 implies that $J(Q, q, x)$ satisfies a 24-th order linear q-difference equation but this will no play a role in our paper. Of importance is the fact that the 25-th order equation $L_5 f = 0$ has a distinguished 5-dimensional space of solutions, given explicitly by a q-version of the Frobenius method. Since this method is well-known for linear differential equations, but less so for linear q-difference equations, we give more details than usual. For additional discussion on this method, see Wen [40], and for references for the q-gamma and q-beta functions, see De Sole–Kac [10].

First, we define an n-th derivative of an operator $P(E, Q, q)$ by

$$P^{(n)}(E, Q, q) = \sum_{k=0^d} k^n c_k(Q, q) E^k, \qquad P(E, Q, q) = \sum_{k=0^d} c_k(Q, q) E^k.$$

In other words, we may write $P^{(n)} = (E\partial_E)^n(P)$.

Lemma 4.2. *For a linear q-difference operator $P(E, Q, q)$ we have*

$$P(e^x E, Q, q) = \sum_{n=0}^{\infty} \frac{x^n}{n!} P^{(n)}(E, Q, q). \tag{4.5}$$

Moreover, for all natural numbers n and a function $f(Q, q)$ we have

$$P((\log Q)^n f) = \sum_{k=0}^{n} \binom{n}{k} (\log Q)^{n-k} (\log q)^k P^{(n-k)} f. \tag{4.6}$$

Proof. Equations (4.5) and (4.6) are additive in P, hence it suffices to prove them when $P = E^a$ for a natural number a, in which case $(E^a)^{(n)} = a^n$ and both identities are clear. ∎

Lemma 4.3. *Suppose $P(E, Q, q)$ is a linear q-difference operators with coefficients polynomials in E and Q, and $J(Q, q, x) \in \mathbb{Q}(q)[[Q, x]]$ is such that*

$$P(e^x E, Q, q) J(Q, q, x) = O(x^{N+1}) \tag{4.7}$$

for some natural number N. Then,

$$\sum_{k=0}^{n} \binom{n}{k} P^{(k)} J_{n-k} = 0 \tag{4.8}$$

for $n = 0, \ldots, N$, where $J_k = \mathrm{coeff}\big(J(Q, q, x), x^k\big)$, and

$$P f_n = 0, \qquad f_n = \sum_{k=0}^{n} \binom{n}{k} (\log Q)^{n-k} (\log q)^k J_k \tag{4.9}$$

for $n = 0, 1, \ldots, N$. In other words, the equation $Pf = 0$ has $N+1$ distinguished solutions given by

$$f_0 = J_0,$$
$$f_1 = \log Q J_0 + \log q J_1,$$
$$f_2 = (\log Q)^2 J_0 + 2 \log Q \log q J_1 + (\log q)^2 J_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Proof. Equation (4.8) follows easily using (4.5) and by expanding the left hand side of equation (4.7) into power series in x and equating the coefficient of x^n with zero for $n = 0, 1, \ldots, N$. Equation (4.9) follows from equations (4.8) and (4.6), and induction on n. ∎

5   Quantum K-theory versus Chern–Simons theory

There are several hints in the physics literature pointing to a deeper relation between Quantum K-theory and Chern–Simons gauge theory (e.g., for 3-manifolds with boundary, such as knot complements), see for instance in [6, 13, 15, 29, 30] and in references therein. In this section we discuss and comment on the q-difference equations in Chern–Simons theory, gauged linear σ-models and Quantum K-theory. We will discuss three aspects of this comparison:

  (a) q-holonomic systems and their $q = 1$ semiclassical limits,

  (b) ε-deformations,

  (c) matrix-valued invariants.

We begin with the case of the Chern–Simons theory. The partition function of Chern–Simons theory with compact (e.g., SU(2)) gauge group on a 3-manifold (with perhaps nonempty boundary) is given by a finite-dimensional state-sum whose summand has as a building block the quantum n-factorial. This follows from existence of an underlying TQFT [36, 39, 42] which reduces the computation of the partition function into elementary pieces. For the complement of a knot K in S^3, the partition function recovers the colored Jones polynomial of a knot which, in the case of SU(2), is a sequence $J_{K,n}(q) \in \mathbb{Z}[q^{\pm}]$ of Laurent polynomials which can be presented as a finite-dimensional sum whose summand has as a building block the finite q-Pochhammer symbol $(q; q)_n$. This ultimately boils down to the entries of the R-matrix which are given for example in [36].

On the other hand, Chern–Simons theory with complex (e.g., $\mathrm{SL}_2(\mathbb{C})$) gauge group is not well-understood as a TQFT. However, the partition function for a 3-manifold with boundary can be computed by a finite-dimensional state-integral whose integrand has as a building block Faddeev's quantum dilogarithm function [16]. The latter is a ratio of two infinite Pochhammer symbols which form a quasi-periodic function with two quasi periods. (Recall that the Pochhammer symbol is $(x; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j x)$.) These are the state-integrals studied in quantum Teichmüller theory by Kashaev et al. [3, 4, 31] and in complex Chern–Simons theory by Dimofte et al. [11, 12].

The appearance of q-holonomic systems in Chern–Simons theory with compact/complex gauge group is a consequence of Zeilberger theory [35, 41, 43] applied to finite-dimensional state-sums/integrals whose summand/integrand has as a building block the finite/infinite q-Pochhammer symbol. This is exactly how it was deduced that the sequence of colored Jones polynomials $J_{K,n}(q)$ of a knot satisfy a linear q-difference equation $A_K(\hat{L}, \hat{M}, q)J_K = 0$ (see [19]), where $\hat{L}$ and $\hat{M}$ are q-commuting operators that act on a sequence $f \colon \mathbb{N} \to \mathbb{Q}(q)$ by

$$\big(\hat{L}f\big)(n) = f(n + 1), \qquad \big(\hat{M}f\big)(n) = q^n f(n), \qquad LM = qML.$$

In the case of state-integrals, the existence of two quasi-periods leads to a linear q- (and also $\tilde{q}$-)difference equation, where $q = \mathrm{e}^{2\pi \mathrm{i} h}$ and $\tilde{q} = \mathrm{e}^{-2\pi \mathrm{i}/h}$.

It is conjectured that the linear q-difference equation of the colored Jones polynomial essentially coincides with the one of the state-integral, and that the classical $q = 1$ limit (the so-called AJ conjecture [17]) coincides with the A-polynomial $A_K(L, M, 1)$ of the knot. The latter is the $\mathrm{SL}_2(\mathbb{C})$-character variety of the fundamental group of the knot complement, viewed from the boundary torus [8]. Finally, the semiclassical limit (the analogue of (1.17) is given by

$$A_K\big(\hat{L}, \hat{M}, q\big) = A_K(L, M, 1) + D_K(z, \partial_z)h^s + O\big(h^{s+1}\big),$$

where $D_K(z, \partial_z)$ is a linear differential operator of degree s where s is the order of vanishing of $A_K(L, 1, 1)$ at $L = 1$. This order is typically 1 (e.g., for the 4_1, 5_2, 6_1 and more generally all twist knots) but it is equal to 2 for the 8_{18} knot.

We now come to the feature, namely an expected "factorization" of state-integrals into a finite sum of products of q-series and $\tilde{q}$-series. This factorization is computed by an ε-deformation of q- and $\tilde{q}$-hypergeometric series that arise by applying the residue theorem to the state-integrals. For a detailed illustration of this, we refer the reader to [6, 18] and [21].

Our last discussed feature, namely a matrix-valued extension of the Chern–Simons invariants with compact/complex gauge group was recently discovered in two papers [20, 21]. More precisely, it was conjectured and in some cases verified that the scalar valued quantum knot invariants such as the Kashaev invariant [32] (an evaluation of the n-th colored Jones polynomial at n-th roots of unity) and the Andersen–Kashaev state-integral [3] admit an extension into a matrix-valued invariants. The rows and columns are labeled by the set $\mathcal{P}_M$ of $\mathrm{SL}_2(\mathbb{C})$ boundary-parabolic representations of $\pi_1(M)$. In the case of a knot complement, the set $\mathcal{P}_M$ can be thought of as the set of branches of the A-polynomial curve above a point (where the meridian has eigenvalues 1). Although the corresponding vector space $R(M) := \mathbb{Q}\mathcal{P}_M$ with basis $\mathcal{P}_M$ has no ring structure known to us, it has a distinguished element corresponding to the trivial $\mathrm{SL}_2(\mathbb{C})$-representation that plays an important role. A ring structure $\mathbb{Q}\mathcal{P}_M$ might be defined as the Grothendieck group of an appropriate category associated to flat connections on 3-manifolds with boundary, or perhaps by contructing an appropriate logarithmic conformal field theory use fusion rules will define the sought ring as suggested by Gukov. Alternatively, the sought ring might be described in terms of $\mathrm{SL}(2,\mathbb{C})$-Floer homology, suggested by Witten. Alternatively, it might be described by the quantum K-theory of the mirror of the local Calabi–Yau manifold $uv = A_M(x, y)$, (where A_M is the A-polynomial discussed above), suggested by Aganagic–Vafa [1].

We now discuss the above features (a)–(c) that appear in the 3d-gauged linear σ-models and their 3d-3d correspondence studied in detail in [6, 13, 14, 15, 29, 30] and references therein. The q-holonomic aspect is still present since the (so-called vortex) partition function is a finite-dimensional integral whose integrand has as a building block the infinite Pochhammer symbol (note however that $\tilde{q}$ does not appear). The second aspect involving ε-deformations is also present for the same reason as in Chern–Simons theory. The third aspect is absent in general.

We finally discuss the above features in genus 0 quantum K-theory of the quintic. The first aspect is different: the linear q-differential equation has coefficients which are analytic (and not polynomial) functions of Q and q. The classical limit $q = 1$ of the linear q-difference equation of the quintic is given by $\gamma(Q, 1)^{-1}\Delta^4$ (1.17) and this defines a degenerate analytic curve in $\mathbb{C} \times \mathbb{C}$ that consists of a finite collection of lines with coordinates (Δ, Q). On the other hand, the semi-classical limit (i.e., the coefficient of h^4 in (1.17)) is the famous Picard–Fuchs equation of the quintic. The second feature, the ε-deformation for a nilpotent variable ε is encoded in the fact that $K(X)$ has nilpotent elements x. The last feature is most interesting since the matrix-valued invariants are encoded in $\mathrm{End}(K(X))$, where $K(X)$ is not just a rational vector space, but a ring unit 1. It follows that the linear q-difference equations have not only a distinguished solution $J_X(Q, q, 0)$ but a basis of solutions parametrized by a basis $\{\Phi_\alpha\}$ of $K(X)$.

Let us end our discussion with some questions on the colored Jones polynomial $J_{K,n}(q)$ colored by the n-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$ representation. For simplicity, we abbreviate $R(S^3 \setminus K)$ defined above by $R(K)$.

Question 5.1.

(a) *Does the vector space $R(K)$ have a ring structure?*

(b) *If so, is the series $\sum_{n=1}^{\infty} J_{K,n}(q)Q^n$ the coefficient of 1 in the $R(K)$-valued small J-function $J_K(Q, q, 0)$ of a knot K?*

(c) *If so, is there a t-deformation $J_K(Q, q, t)$?*

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