Pathlike Co/Bialgebras and their Antipodes with Applications to Bi- and Hopf Algebras Appearing in Topology, Number Theory and Physics

Ralph M. KAUFMANN [ab] *and Yang MO* [a]

a) *Department of Mathematics, Purdue University, West Lafayette, IN, USA*
   E-mail: *rkaufman@purdue.edu*, *mo12@purdue.edu*
   URL: https://www.math.purdue.edu/~rkaufman/

b) *Department of Physics and Astronomy, Purdue University, West Lafayette, IN, USA*

Abstract. We develop an algebraic theory of colored, semigrouplike-flavored and pathlike co-, bi- and Hopf algebras. This is the right framework in which to discuss antipodes for bialgebras naturally appearing in combinatorics, topology, number theory and physics. In particular, we can precisely give conditions for the invertibility of characters that is needed for renormalization in the formulation of Connes and Kreimer. These are met in the relevant examples. In order to construct antipodes, we discuss formal localization constructions and quantum deformations. These allow to define and explain the appearance of Brown style coactions. Using previous results, we can interpret all the relevant coalgebras as stemming from a categorical construction, tie the bialgebra structures to Feynman categories, and apply the developed theory in this setting.

Key words: Feynman category; bialgebra; Hopf algebra; antipodes; renomalization; characters; combinatorial coalgebra; graphs; trees; Rota–Baxter; colored structures

2020 Mathematics Subject Classification: 16T05; 18M85; 81T15; 81R50

Dedicated to Professor Dirk Kreimer
on the occasion of his 60*th birthday*

1 Introduction

Although the use of Hopf algebras has a long history, the seminal paper [13] led to a turbocharged development for their use which has penetrated deeply into mathematical physics, number theory and also topology, their original realm – see [1] for the early history. The important realization in [13, 14, 15] was that the renormalization procedure in quantum field theory can be based on a character theory for Hopf algebras via the so-called Birkhoff factorization and convolution inverses. The relevant Hopf algebras are those of trees with a coproduct given by a sum over so-called admissible cuts – with the factors of the coproduct being the left-over stump and the collection of cut-off branches – and Hopf algebras of graphs in which the factors of the summands of the coproduct are given by a subgraph and the quotient graph. A planar version of the tree formalism was pioneered in [22], see also [54]. The appearance in number theory of this type of coproduct goes back to [27]. It was developed further and applied with great success in [7]. In all these cases, the product structure is free and the coproduct is the carrier of information. The group of characters was previously studied in [9]. A precursor to the

Hopf algebraic considerations can be found in [4]. In [24, 25], we gave the details to prove the results announced in [41] that all these structures stem from a categorical framework, where the coproduct corresponds to deconcatenation of morphisms. Such coproduct structures can be traced back to [47] and appear in combinatorics [29, 58]. Dual algebra structures can be found in [57] and [18].[1] In [41], we developed a theory of so-called Feynman categories, which are essentially a special type of monoidal category and could connect the product structure to the monoidal structure on the level of morphisms. Special cases are related to operads, the simplicial category and categories whose morphisms are classified by graphs. In [25], we could show that the monoidal product in these categories is compatible with the deconcatenation co-product thus yielding bialgebras. This generalizes those of Baues and Goncharov [4, 27] which are simplicial in nature, the Connes–Kreimer tree Hopf algebras, which have operadic origin [24], and the Connes–Kreimer Hopf algebras of graphs as well as the core Hopf algebra [44], which are graphical and more categorical in nature.

The co- and bialgebras which have a categorical interpretation also include path coalgebras and incidence coalgebras, as for instance considered in [29, 58], see Section 2.2.1. There are two versions of the story, one is symmetric and yields cocommutative bialgebras and the second is non-symmetric and in general yields to non-cocommutative bialgebras, such as those from planar structures. In all the examples one passes from the bialgebras to a connected quotient to obtain an antipode by invoking Quillen's formalism [56].

In this paper, we address the question of obstructions to constructing antipodes on the bialgebra level. The questions that we can now answer is:

> What structures precisely exist already on the bialgebra level, and what is their role in the inversion of characters and the construction of antipodes?

To this end, we give an *algebraic characterization* of a classes of bialgebras amenable to such considerations. These are *colored*, *sg-flavored* (sg stands for semigrouplike), and *pathlike* coalgebras. They capture and generalize the essential features of categorical coalgebras, whose paradigmatic example are path algebras for quivers. This allows us to specify effective criteria for the existence of antipodes and convolution invertibility of characters. We also work over an arbitrary commutative ground ring. The origins for the theory of sg-flavored coalgebras can be traced back to [59] and with hindsight, several of the quotient construction are foreshadowed in [58] for the special case of incidence bialgebras.[2] The existence of antipodes is usually often established via some sort of connectivity of a filtration [56, 60]. The problem is that the bialgebras in question are usually not connected for the standard filtrations, which is why quotients are taken.

Concretely, after reviewing some basic structures in the generality we need in Section 2.1, providing a short list of paradigmatic examples to illustrate and motivate the constructions Section 2.2, and touching upon the complications from isomorphisms, we introduce the first key notion, that of colored coalgebras in Section 2.3. This allows us to generalize Quillen's connected coalgebras and conilpotent coalgebras to many colors. In the categorical setting, the colors are the objects and algebraically speaking they correspond to grouplike elements. The main structural result is Theorem 2.18. The basic examples like path algebras are of this type, and appear as the dual coalgebras $C[M]$ to colored monoids M. Proposition 2.24 classifies their properties and gives criteria for color connectedness. Dualities in a general setting are tricky; we provide some technical details and the dual notion of colored algebras in Appendix A.

To construct antipodes and convolution inverses, we introduce the fundamental concept of a QT-filtration (Quillen–Takeuchi) in Section 3.1, which generalizes the commonly used filtrations. Such a filtration is called a sequence if it is exhaustive. This is what is needed to make Takeuchi's argument work and results in:

---

[1] We thank a referee for pointing out this reference.

[2] We thank D. Kreimer for pointing this out.

Theorem 3.3. *Consider a coalgebra C with counit ϵ and an algebra A with unit η. Given a QT-sequence $\{F_i\}_{i=0}^{\infty}$ on C, such that $\mathrm{Ext}_R^1(C/F_0C, A) = 0$, for any element f of $\mathrm{Hom}(C, A)$, f is $\star$-invertible if and only if the restriction $f|_{F_0C}$ is $\star$-invertible in $\mathrm{Hom}(F_0C, A)$.*

A special type of QT-filtration, which we call the bivariate Quillen filtration (see Section 3.2), arises by considering skew primitive elements. These are the essential actors for so-called sg-flavored coalgebras and are key to understanding the structures of the bialgebras before taking quotients. In the applications a slightly stronger condition is satisfied for the level 0 part of the filtration, which leads to the central notion of pathlike co/bialgebras, cf. Section 3.3.

Theorem 3.21. *Let C be pathlike coalgebra and A algebra with $\mathrm{Ext}\left(C/F_0^{Q\bullet\bullet}C, A\right) = 0$. An element $f \in \mathrm{Hom}(C, A)$ has an $\star$-inverse in the convolution algebra $\mathrm{Hom}(C, A)$ if and only if for every grouplike element g, $f(g)$ has an inverse as ring element in A.*

In particular, in this situation, a character is $\star$-invertible if and only if it is grouplike invertible.

Furthermore, a pathlike bialgebra B is a Hopf algebra if and only if the set of grouplike elements form a group.

Color connected coalgebras are pathlike and hence these general results apply to them.

These results are then directly applicable to renormalization via Rota–Baxter algebras and Birkhoff decompositions, which are briefly reviewed in Section 4.1, with more details in Appendix B. A closer inspection of this framework yields two paths of actions. The first is to realize that for invertibility of specific characaters the bialgebras need not necessarily be Hopf algebras. Thus limiting the characters by imposing certain restrictions on them or the target algebra will ensure their invertibility in the pathlike case. The second avenue is to formally invert the grouplike elements. Combining the two approaches yields several universal constructions, through which characters with special properties factor, cf. Section 4. This for instance naturally leads to *quantum deformations* of the algebras. These results generalize and explain similar constructions in [24]. They allow us to construct Brown style coactions [7], see Proposition 3.18. Applying this to pathlike bialgebras, viz. bialgebras whose coalgebras are pathlike yields:

Theorem 4.14. *Let B be a pathlike bialgebra: Every grouplike invertible character has a $\star$-inverse and vice-versa.*

The quotient bialgebra B/I_N of Proposition 4.2 is connected and hence Hopf.

In particular, every grouplike normalized character $\phi \in \mathrm{Hom}(B, A)$, when factored through to $\bar{\phi} \in \mathrm{Hom}(B/I_N, A)$, has an inverse computed by $\bar{\phi}^{-1} = \bar{\phi} \circ S$.

The bialgebra $B_{\mathbf{q}}/I$ is Hopf and the $\star$ inverse of a grouplike central character $\phi \in \mathrm{Hom}(B, A)$, when lifted and factored through $\bar{\bar{\phi}} \in \mathrm{Hom}(B_{\mathbf{q}}/I, A)$, has an inverse computed by $\bar{\bar{\phi}}^{-1} = \bar{\bar{\phi}} \circ S$.

Let $S = \mathrm{gpl}(B)$, viz. the grouplike elements, then the left localization BS^{-1} is a Hopf algebra. Moreover if S is satisfies the Ore condition and is cancellable, there is an injection $B \to BS^{-1}$ and $B_q/I = BS^{-1}/J$, where J is the ideal generated by $(bg - gb)$ for all $b \in B$, $g \in \mathrm{gpl}(B)$.

This framework is applicable to bialgebras arising from Feynman categories, as we show in Section 5, resulting in the main structural Theorem 5.7 in the non-symmetric case and Theorem 5.9 in the symmetric case. We concretely apply this to the classes of Feynman categories, set based or simplicial, (co)operadic and graphical corresponding to the examples mentioned in the beginning. To be more self-contained, these are given in a new set-theoretical presentation in an Appendix C.

We end in Section 6 with a conclusion and an outlook, which in particular discusses strategies going beyond the pathlike setting by using double categories and Drinfel'd doubles, cf. Section 6.2.

2   Colored algebras and coalgebras

Before delving into the constructions, we review the general notions, recall the key examples and introduce the notion of colored coalgebras, which formalizes the type of coalgebra obtained from a category or equivalently a partial monoid that is colored. We end with a short overview of the complications that arise in the presence of isomorphisms. One resolution to this problem is to work with equivalence classes, which is done in Section 5.

2.1   Setup and basic notation

We provide some of the relevant notions and refer to [10, 53] for more details. Throughout we work over a commutative unital ground ring R. If this is taken to be a field, then it is denoted by k. We will denote the free R-module on a set X by $R^{(X)} = \bigoplus_{x \in X} R$. Tensors are understood to be over the ground ring, thus $\otimes$ means $\otimes_R$. For an R-module M there is a canonical isomorphism $M \otimes R \simeq M$ given by the R-module structure on M, and we will simply implement this isomorphism as an identification-effectively making R a strict monoidal unit. We will either work in the category of R-modules *or graded* R-modules. In the second case, we assume that R is concentrated in degree 0 and the grading is by $\mathbb{N}$ unless otherwise stated. We will assume that algebras have a unit and coalgebras have a counit unless otherwise stated. A graded coalgebra is connected if $C_0 = R$.

2.1.1   Internal product

There are several complications when working over a commutative ring. First, there is no canonical way to identify a tensor product of submodules with a submodule. This may lead to different subcoalgebra structures. Second, a submodule with the restriction of the coalgebra structure may not yield a coalgebra. It may not even be possible to give a coalgebra structure. To avoid these issues, we will use an internal product denoted by $\boxtimes$ for the underlying R-submodules.

Given a coalgebra (C, Δ, η) over R and two submodules M, N of C, their internal product $M \boxtimes N$ is defined to be the following submodule of $C \otimes C$:

$$M \boxtimes N := \{\text{finite sums of } \ x_{(1)} \otimes x_{(2)} \in C \otimes C \,|\, x_{(1)} \in M, \, x_{(2)} \in N\}.$$

For two families of subcomodules $\{N_j\}$, $\{M_i\}$, we will also use the notation $\Delta(x) \in \sum_{i,j} M_i \boxtimes N_j$. This means $\Delta(x)$ can be decomposed as a finite sum of $x_{(1)} \otimes x_{(2)}$ such that every summand belongs to $M_i \boxtimes N_j$ for some i, j. If $M \subset M'$ and $N \subset N'$ as submodules, then $M \boxtimes N \subset M' \boxtimes N'$. For a submodule S, we say that V *is a subcoalgebra*, if $\Delta(S) \subset S \boxtimes S$. Note that the counit $\epsilon|_S \colon S \to R$ satisfies the equations of a counit.

Example 2.1 ([55])**.** Consider the $\mathbb{Z}$-module $C = \frac{\mathbb{Z}}{8\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. Let $x = (1,0)$ and $z = (0,1)$ and endow C with a coalgebra structure by setting $\Delta(x) = 0$ and $\Delta(z) = 4x \otimes x$. Since $4x \otimes x$ has order of 2 in $C \otimes C$ the coproduct is well-defined. Also, $\epsilon(x) = \epsilon(z) = 0$, because they are torsion elements. Let $y = 2x$ and consider $V = \mathbb{Z}y + \mathbb{Z}z \subset C$. As $\Delta(z) = y \otimes y$, $\Delta(V) \subset V \boxtimes V$. But, there is no coalgebra structure on V such that the natural inclusion is a coalgebra morphism. Indeed, the map $\Delta \colon V \to C \otimes C$ has no lifting to $V \otimes V$ because every preimage of $\Delta(z)$ has order 4.

The complication arises due to torsion elements, in the case that $R = k$ is a field, we can work with the usual monoidal product $\otimes$.

2.1.2 Semigrouplikes and connected coalgebras

Given a coalgebra C, an element g in C is said to be *semigrouplike* if $\Delta(g) = g \otimes g$. If additionally $\epsilon(g) = 1$ then g is said to be *grouplike*. Note that 0 is always semi-grouplike, but never grouplike. The set of *non-zero* semigrouplike elements in C is denoted by $\mathrm{sgpl}(C)$ and the set of grouplike elements in C by $\mathrm{gpl}(C)$. If the coalgebra has a counit ϵ, then for $g \in \mathrm{sgpl}(G)$: $(1-\epsilon(g))g = 0$ that is $1 - \epsilon(g) \in \mathrm{Ann}_R(g)$. Hence such a g is a torsion element, if it is not grouplike. Let g and h be grouplike elements in C, an element x in C is called (g,h)-*skew primitive* if $\Delta(x) = x \otimes g + h \otimes x$. When $g = h$, it is called g-*primitive element*.

In a coaugmented coalgebra, the coaugmentation $\eta \colon R \to C$ preserves the counit. Thus, setting $e = \eta(1)$, $\Delta(e) = e \otimes e$, there is a splitting $C = \mathrm{Re} \oplus \overline{C}$ and $\overline{C} = \ker(\epsilon)$. The reduced diagonal $\overline{\Delta}$ is defined by $\overline{\Delta}(x) = \Delta(x) - e \otimes x - x \otimes e$. In a graded coalgebra all grouplike elements are necessarily in degree 0 as $2i = i$ implies $i = 0$. In a connected graded coalgebra therefore e, which is the generator of C_0, is the unique grouplike element.

In a bialgebra, the semigrouplike elements form a monoid: for two semigrouplike elements g, h: $\Delta(gh) = gh \otimes gh$. The grouplike elements form a submonoid: $\epsilon(gh) = \epsilon(g)\epsilon(h) = 1$. If C is coaugmented, these structures are unital with unit e.

In a Hopf algebra any semigrouplike element g satisfies $S(g)g = gS(g) = \epsilon(g)e$. In particular, if an element is grouplike, $S(g) = g^{-1}$ is the only possible value for an antipode.

2.1.3 Convolution algebra and characters

Recall that given a coalgebra C and an algebra A, the *convolution algebra* is the R-module $\mathrm{Hom}(C, A)$ with multiplication given by $(f \star g)(x) = m \circ (f \otimes g) \circ \Delta(x) = \sum_x f(x_{(1)})g(x_{(2)})$. If A is unital with unit η and C is counital with counit ϵ then $\eta \circ \epsilon$ is a unit for the convolution algebra.

Lemma 2.2. *Given a subcoalgebra S, $\mathrm{Hom}(S, A)$ is a unital subalgebra of the convolution algebra with unit $(\eta \otimes \epsilon)|_S = \eta \circ \epsilon|_S$.*

Proof. One has to check that $\mathrm{Hom}(S, C)$ is closed under composition and that it contains the unit. Since $\Delta(S) \subset S \boxtimes S$, the formula $(f \star g)(s) = m(f \otimes g) \circ \Delta(s) = \sum f(s_{(1)})g(s_{(2)})$ yields a function from $S \boxtimes S \to C$. The unit restricts appropriately. ∎

Lemma 2.3. *If C is coaugmented and A is augmented, then if f preserves the augmentation or the coaugmentation, so does its convolution inverse.*

Proof. Assume $f(1_C) = 1_A$, we need to show $g(1_C) = 1_A$. On one hand

$$(f \star g)(1_C) = m_A \circ (f \otimes g) \circ \Delta_C(1_C) = (m_A \circ (f \otimes g))(1_C \otimes 1_C) = f(1_C)g(1_C) = g(1_C).$$

On the other hand, $(f \star g)(1_C) = \eta_A \epsilon_C(1_C) = 1_A$. The compatibility with the counit is established by:

$$\epsilon_C(a) = \epsilon_C(a)1_R = \epsilon_C(a)\epsilon_A(1_A) = \epsilon_A(\epsilon_C(a)1_A) = \epsilon_A(\epsilon_C(a)\eta_A(1_A)) = \epsilon_A(\eta_A(\epsilon_C(a)))$$

$$= \epsilon_A\left(\sum_a f(a_{(1)})g(a_{(2)})\right) = \sum_a \epsilon_A(f(a_{(1)}))\epsilon_A(g(a_{(2)})) = \sum_a \epsilon_C(a_{(1)})\epsilon_A(g(a_{(2)}))$$

$$\times \epsilon_A\left(\sum_a \epsilon_C(a_{(1)})g(a_{(2)})\right) = \epsilon_A\left(g\left(\sum_a \epsilon_C(a_{(1)})a_{(2)}\right)\right) = \epsilon_A(g(a)). \qquad \blacksquare$$

An important fact is that an antipode S for a Hopf algebra H is a convolution inverse to $\mathrm{id} \in \mathrm{Hom}(H, H)$, where on the left the coalgebra structure of H is used and on the right, the algebra structure of H is used. As id preserves the coaugmentation and augmentation, so does S.

If B is a bialgebra over R and A is algebra over R, then the set of *characters* of B with values in A are the algebra homorphisms $\mathrm{Hom}_{R-\mathrm{alg}}(B, A)$.

Definition 2.4. We say a character is *grouplike invertible*, if for every grouplike element $g\colon \phi(g) \in A^{\times}$, *grouplike central* if for all $g \in \mathrm{gpl}(C)\colon \phi(g) \in Z(A)$, grouplike scalar if $\phi(g) \in R^{\times}$ and grouplike normalized if $\phi(g) = 1$ for all grouplike g.

Recall that in a graded setting, the grouplike elements are in degree 0, and if A is graded and ϕ preserves the grading, then the grouplike elements have to land in A_0. It is quite common that, even if A is not commutative, A_0 is. In this case all the characters preserving the grading will be grouplike central. In several applications, the characters are scalar and take values $\left(\frac{1}{2\pi i}\right)^{k}$, which in turn is also a form of grading, cf. Section 4.2.

Lemma 2.5. *Any $\star$-invertible character is grouplike invertible.*

Proof. This follows from the fact that for any grouplike $\phi^{\star-1} \star \phi(g) = \phi^{\star-1}(g)\phi(g) = 1_B$. ∎

The converse of this is true under a specific conditions on A for pathlike bialgebras, see Theorem 3.21 and Section 4.

Proposition 2.6. *If A is commutative, the characters form an algebra under convolution.*
If B is a Hopf algebra, then $\phi \circ S = \phi^{-1}$ is the convolution inverse.
If B is Hopf and A is commutative, then the characters form a group.

Proof. For the first statement:

$$(f \star g)(ab) = (f \otimes g)\left(\sum a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}\right) = \sum f(a_{(1)})f(b_{(1)})g(a_{(2)})g(b_{(2)})$$
$$= \sum f(a_{(1)})g(a_{(2)})f(b_{(1)})g(b_{(2)}) = (f \star g)(a)(f \star g)(b).$$

For the second statement:

$$((\phi \circ S) \star \phi)(a) = \sum_a \phi(S(a_{(1)}))\phi(a_{(2)}) = \sum_a \phi(a_{(1)}S(a_{(2)})) = \phi(\eta \circ \epsilon(a)) = (\eta \circ \epsilon)(a).$$

For the last statement, we need to show that the inverse of a character is a character. Indeed, $\phi^{-1}(ab) = \phi(S(ab)) = \phi(S(b)S(a)) = \phi(S(b)\phi(S(a)) = \phi(S(a))\phi(S(b))$, where the last equation holds since A is commutative. ∎

2.2 Key examples of coalgebras and complications

2.2.1 Path coalgebra

The paradigmatic example and namesake for the article is the *path coalgebra* of a quiver. Given a quiver Q, that is a graph with directed edges, a path is given by a sequence of consecutive directed edges $p = (\vec{e}_1 \cdots \vec{e}_n)$. Consecutive means that the target vertex $t(\vec{e}_i)$ of $\vec{e}_i$ is the source vertex $s(\vec{e}_{i+1})$ of $\vec{e}_{i+1}$. By definition for each vertex v there is an empty path of length 0, with source and target v. This the identity at v, which is traditionally denoted simply by v. Using this notation, let $P(Q)$ be the set of paths of Q, then $R^{(P(Q))}$ has the coalgebra structure

$$\Delta(\vec{e}_1 \cdots \vec{e}_n) = s(\vec{e}_1) \otimes (\vec{e}_1 \cdots \vec{e}_n) + \sum_{i=1}^{n-1}(\vec{e}_1 \cdots \vec{e}_i) \otimes (\vec{e}_{i+1} \cdots \vec{e}_n) + (\vec{e}_1 \cdots \vec{e}_n) \otimes t(\vec{e}_n).$$

The counit is $\epsilon(v) = 1$ and $\epsilon(\vec{e}_1 \cdots \vec{e}_n) = 0, n \geqslant 0$. There is a grading given by the length of a path. The grouplike elements are exactly the length 0 paths v. The paths of length 1 are exactly the skew primitive, with $(\vec{e})$ being $(s(\vec{e}), t(\vec{e}))$-skew primitive. This coalgebra is not connected if there is more than one vertex.

Remark 2.7. A special important case arises if one considers the quiver for a complete graph, that is one directed edge per pair of vertices. A path is then equivalently given by a sequence of vertices, that is simply a *word in vertices*. In particular, the complete graph on two vertices $\{0, 1\}$, yields the quiver $\circlearrowleft 0 \leftrightarrows 1 \circlearrowright$ whose path algebra underlies Goncharov's and Brown's Hopf algebras for polyzetas [7, 27]. This is also the fundamental path groupoid for $\mathbb{C} \backslash \{0, 1\}$ with tangential base points and is directly linked, cf. [24, Section 1], to Chen iterated integrals [12]. The case with many vertices corresponds to polylogs. The bi-algebra structure is actually founded in a simplicial structure, see [24, Section 4], [25, Section 3.3.1].

2.2.2 Incidence coalgebra

Another big source of inspiration and examples comes from combinatorics via *incidence coalgebras*, as studied in [29, 58]. This is a coalgebra on the free R module on the set of intervals $[x, y]$

$$\Delta([x, y]) = [x, x] \otimes [x, y] + \sum_{z \colon x < z < y} [x, z] \otimes [z, y] + [x, y] \otimes [y, y].$$

The co-unit evaluates to 1 on $[x, x]$ and 0 else.

2.2.3 Categorical coalgebra

These two examples are special cases of coalgebras stemming from a category with finite decomposition, which will be the main case of interest in the applications. Let $\mathcal{C}$ be a small category. This means that both the objects $X = \mathrm{Obj}(\mathcal{C})$ and the morphisms $M = \mathrm{Mor}(\mathcal{C})$ are sets. The mapping x to id_x identifies X with a subset $X \subset M$ and $M = X \amalg \overline{M}$. The free R modules split accordingly $R^{(M)} = R^{(I)} \oplus R^{(\overline{M})}$.

Furthermore, restrict to the case where $\mathcal{C}$ is decomposition finite, this is for each $\phi \in M$, there are only finitely many pairs $(\phi_0, \phi_1) \in M \times M$ such that $\phi_0 \cdot \phi_1 = \phi_1 \circ \phi_0 = \phi$.

The *categorical monoid coalgebra* $C[M]$ is defined to be the free R-module $R^{(M)}$ with coproduct for $\phi \colon x \to y$ given by

$$\Delta(\phi) = \sum_{(\phi_0, \phi_1) \colon \phi_0 \cdot \phi_1 = \phi} \phi_0 \otimes \phi_1 = \mathrm{id}_x \otimes \phi + \phi \otimes \mathrm{id}_y + \sum_{(\phi_0, \phi_1) \in \overline{M} \times \overline{M} \colon \phi_0 \cdot \phi_1 = \phi} \phi_0 \otimes \phi_1$$

and counit $\epsilon(\mathrm{id}_x) = 1$ for the identity maps and $\epsilon(\phi) = 0$ if ϕ is not an identity map. So, letting ϵ_X be the projection onto $R^{(X)}$, ϵ factors through ϵ_X.

These coalgebras were analyzed in [24, 25, 41], and their history goes back to [47], see also [29].

Remark 2.8. Note that there are two equivalent, opposite, ways to write down the composition maps, which correspond to the two ways to write the compositions $\mathrm{Mor}\,_s\times_t \mathrm{Mor} \to \mathrm{Mor}$ or $\mathrm{Mor}\,_t\times_s \mathrm{Mor} \to \mathrm{Mor}$

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z), \qquad (\phi_0, \phi_1) \mapsto \phi_0 \cdot \phi_1 := \phi_1 \circ \phi_0, \tag{2.1}$$

$$\mathrm{Hom}(Z, Y) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z), \qquad (\phi_0, \phi_1) \mapsto \phi_0 \circ \phi_1. \tag{2.2}$$

We will call the first monoidal and the second categorical. The two coproducts are opposites, i.e., Δ and $\Delta^{\mathrm{op}} = \tau \circ \Delta$, where τ is the flip. The first version of this for the coproduct is what is used for posets and quivers and fits with the Connes–Kreimer coproduct [13, 24, 25], where the subgraph is on the left and the cograph is on the right, cf. (5.3). The second one is more natural from a category point of view and corresponds to the tree Hopf algebra, where the stump is on the left and the branches on the right, cf. (5.2).

Note that this ambiguity is non-essential for a coalgebra or a bialgebra. For a Hopf algebra, it might *a priori* make a difference, – $H^{\mathrm{op,cop}}$ is a Hopf algebra, but H^{cop} may not be – but if the Hopf algebra comes from a category H^{cop}, is also Hopf, cf. Proposition 3.23.

Remark 2.9. Path and incidence coalgebras arise from categories as follows: For $P(Q)$ the objects are the vertices of Q and $M = P(Q)$. The source s and target map t map a path to its start and end vertices, respectively. The composition is the concatenation of paths, and the identities are the length 0 paths. This is the free category generated by the morphisms corresponding to the edges.

A poset $(S, \prec)$ defines a category whose objects are S and whose morphisms are defined as follows. There is one exactly one morphism $\phi_{x \to y}$ between x and y if and only if $x \leq y$. The category is locally finite when the poset is. The categorical coproduct is $\Delta(\phi_{x \to y}) = \sum_{z \in [x,y]} \phi_{x \to y} \otimes \phi_{y \to z}$ or in the usual poset notation, where one identifies $\phi_{x \to y}$ with the interval $[x, y]$. The identities are $\mathrm{id}_x = [x, x]$.

This category is the quotient of a quiver category. The quiver has vertices S and directed edges given by the (x, y)-skew primitives. These are the elements $x \prec y$, $x \neq y$, where there is no $z \colon x \prec z \prec y$. The quotient is given by identifying two morphisms, i.e., paths, p and q whenever they have the same source and target. Categorically speaking this trivializes π_1 of the category making each component simply connected.

Algebraically speaking, a small category is the same as a colored monoid which is a particular type of partial monoid.

Definition 2.10. A *colored monoid* M is a set together with the following data. A set X of colors, two morphisms $s, t \colon M \to X$, and a partial product $\circ \colon M_s \times_t M \to M$ which is associative in the sense that $(ab)c$ exists, then so does $a(bc)$ and they coincide.

It is *unital* if there is morphism $\mathrm{id} \colon X \to M$, which is a section of both s and t such that $a \circ \mathrm{id}(t(a)) = \mathrm{id}(s(a)) \circ a = a$. $M = \amalg_{x,y} M_{x,y}$, where $M_{x,y} = \{a \mid s(a) = x, t(a) = y\}$.

A colored monoid is called *decomposition or locally finite*, if all the fibers $\circ^{-1}(m)$ for $m \in M$ are finite.

It is graded if there is a degree function $\deg \colon M \to \mathbb{N}_0$ such that $\deg(ab) = \deg(a) + \deg(b)$. A degree function is proper is $\deg(a) = 0$ if and only if a is invertible.

The equivalence between colored monoids and categories is given by the identifications $X = \mathrm{Obj}\,\mathcal{C}$ and $M = \mathrm{Mor}(\mathcal{C})$ with $M_{x,y} = \mathrm{Hom}(x, y)$.

We will need the following result [25, Lemma 1.11], which rephrased for colored monoids states that:

Lemma 2.11. *In a decomposition finite colored unital monoid any left or right invertible element is invertible.*

A categorical coalgebra is a dual construction to that of a colored monoid. In the case of finite X this is straightforward. If X is not finite there are subtleties which are relegated to Appendix A. The class of the resulting coalgebras is codified as colored coalgebras.

2.2.4 Complications from isomorphisms

A colored unital monoid is a groupoid, if all of its elements are invertible in the colored sense, viz there is a morphism $^{-1} \colon M \to M$, such that $\phi^{-1} \circ \phi = \mathrm{id}_{s(\phi)}$ and $\phi \circ \phi^{-1} = \mathrm{id}_{t(\phi)}$. As a category, this means that all morphisms are isomorphisms. Given a category $\mathcal{C}$, the underlying groupoid $\mathrm{Iso}(\mathcal{C})$ is defined by the objects of $\mathcal{C}$ with only the isomorphisms. This groupoid acts from the left and right on morphisms by conjugation $\phi \to (\sigma \Downarrow \sigma')(\phi) := \sigma' \circ \phi \circ \sigma^{-1}$.

Lemma 2.12. *The product of M being locally finite necessitates that there are only finitely many objects in each isomorphism class and that all automorphism groups are finite.*

An identity morphism id_x is grouplike in $C[M]$, if and only if x is the only element in its isomorphism class and it has no non-trivial automorphisms.

If there is a skew-primitive morphism $\phi: x \to y$, then x and y are the only elements in their isomorphism class and both have no non-trivial automorphisms.

Proof. For an identity id_x the deconcatenation coproduct will have a term $\sigma \otimes \sigma^{-1}$ of any $\sigma \in \mathrm{Iso}(x, -)$, which proves the first two statements.

For a decomposition finite noninvertible $\phi: x \to y$ in the monoidal convention (2.1) the coproduct is

$$\Delta(\phi) = \sum_{\sigma_x \in \mathrm{Iso}(X,-), \sigma_y \in \mathrm{Iso}(Y,-)} \left[\sigma_x \otimes \sigma_x^{-1} \cdot \phi + \phi \cdot \sigma_y \otimes \sigma_y^{-1} \right] + \cdots, \tag{2.3}$$

where the first summands are always present. The element ϕ being skew-primitive means that all the terms except $\phi \otimes \mathrm{id}_y + \mathrm{id}_x \otimes \phi$ are not present, which is the third statement. ∎

We will call a morphism ϕ *essentially indecomposable* if the only decompositions into two factors have at least one factor which is an isomorphism. For an identity these are the terms $\sigma \otimes \sigma^{-1}$, and if ϕ is not an identity, the corresponding terms of $\Delta(\phi)$ are the displayed terms in (2.3).

Taking isomorphism classes will make it possibly to remedy the situation arising from too few grouplikes and skew-primitives in case the action of $\mathrm{Iso}(\mathcal{C})$ behaves nicely; which it does for a Feynman category, see Section 5.2.

If there is only one object in each isomorphism class of objects, i.e., $\mathcal{C}$ is skeletal, then the sum will only be over automorphisms. In the case of a finite groupoid with just one color, i.e., $M = G$ is a finite group, the deconcatenation coproduct is simply the familiar

$$\Delta(g) = \sum_{h \in G} gh \otimes h^{-1} = \sum_{h \in G} h \otimes h^{-1} g.$$

2.3 Colored co- and bialgebras

2.3.1 Colored coalgebras

The coalgebra on X, denoted by $C[X]$, is the coalgebra structure on $R^{(X)}$, defined by letting all generators be grouplike, i.e., $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$. Such a coalgebra, viz. a coalgebra freely generated by grouplikes, is often called *setlike*.

Lemma 2.13. *A right (or left) coaction ρ by $C[X]$ on an R module M is equivalent to grading by X, that is $M = \bigoplus_{x \in X} M_x$. Similarly having a right and a left coaction is the same as a double grading by X, that is $M = \bigoplus_{(x,y) \in X \times X} M_{x,y}$.*

Proof. Given a coaction, set $M_x = \{m \mid \rho(m) = m \otimes x\}$. Each $\rho(m) = \sum_x m_x \otimes x$ is a finite sum and thus the module is the sum of the M_x. Furthermore $\Delta(x) = x \otimes x$ so that $(m_x)_y = \delta_{x,y} m_x$. and the sum is direct. Vice-versa setting $\rho(m_x) = m \otimes x$ for $m_x \in M_x$ defines a coaction. One can proceed similarly for a left coaction. The fact that left and right coactions commute proves the last statement. ∎

For a coalgebra, given a left comodule N, $\lambda: N \to N \otimes C$ and a right comodule M, $\rho: M \to C \otimes M$ the cotensor product $M_\lambda \square_\rho N$ is defined as the coequalizer

$$M_\lambda \square_\rho N \longrightarrow M \otimes_R N \xrightarrow[\substack{id_M \otimes_R \rho_L}]{\rho_R \otimes_R id_N} M \otimes_R C \otimes_R N.$$

This is generated by elements $c \times c'$ with $\rho(c) = \lambda(c')$ which is dual to the condition that $t(m_1) = s(m_2)$.

Definition 2.14. A *colored coalgebra* C *with colors* X is a coalgebra C together with a bi-co-module structure of C over $C[X]$, that is $\lambda\colon C \to C[X] \otimes C$, $\rho\colon C \to C[X] \otimes C$ such that the comultiplication map is a morphism in the category of $C[X]$ bi-comodules and that Δ factors through the cotensor product:

$$
\begin{array}{ccc}
C & \xrightarrow{\quad\Delta\quad} & C \otimes C \\
& \searrow \qquad \nearrow & \\
& C_\lambda \square_\rho C. &
\end{array}
$$

When omitting the mention of the colors, we implicitly assume the coalgebra is colored by its semigrouplike elements. A colored coalgebra with colors X is *color counital*, if there is a map of $C[X]$ bi-comodule co-algebras $\epsilon_X\colon C \to C[X]$ such that

$$(\epsilon_X \otimes \mathrm{id})\Delta = \lambda \qquad \text{and} \qquad (\mathrm{id} \otimes \epsilon_x)\Delta = \rho,$$

and *color coaugmented* if there is a coalgebra map of $C[X]$ bi-comodules $i_X\colon C[X] \to C$ splitting ϵ_X. A graded color coaugmented coalgebra colored by X is *color reduced* if $C_0 = C[X]$.

By Lemma 2.13 a coalgebra colored by X decomposes as $C = \bigoplus_{(x,y)\in X\times X} C_{x,y}$. The condition of begin a map of $C[X]$ bi-comodules factoring through the cotensor product means that

$$\Delta(C_{x,y}) \subset \bigoplus_z C_{x,z} \boxtimes C_{z,y}.$$

If C is color coaugmented, $C = C[X] \oplus \overline{C}$, where $\overline{C} = \ker(\epsilon_X)$ with $i_X(x) \in C_{x,x}$. Moreover, $C_{x,x} = Ri_X(x) \oplus \overline{C}_{x,x}$. Setting $\overline{C}_{x,y} = C_{x,y}$ this means that $\overline{C} = \bigoplus_{(x,y)\in X\times X} \overline{C}_{x,y}$. Since ϵ_X is a coalgebra map $\epsilon|_{\overline{C}} = 0$.

Remark 2.15. If C is co-augmented as a coalgebra, setting $X = \{e\}$, $\lambda(c) = e\otimes c$ and $\rho(c) = c\otimes e$ defines a coloring with one color $C_{e,e} = C$. Thus, a color coaugmented coalgebra colored by X generalizes the notion of a coaugmented coalgebra to many colors corresponding to sets of grouplike elements.

Definition 2.16. For a color coaugmented coalgebra colored by X, we define the *reduced diagonal* to be $\overline{\Delta} = \Delta - \tilde{\lambda} - \tilde{\rho}$, where $\tilde{\lambda} = (i_X \otimes \mathrm{id})\lambda = i_X\epsilon_X \otimes \Delta$ and $\tilde{\rho} = (\mathrm{id} \otimes i_X)\rho = (\mathrm{id} \otimes i_X\epsilon_X)\Delta$.

Note that $\pi_X = i_X\epsilon_X$ is the projection to the factor $C[X]$.

Definition 2.17. The *Quillen filtration* of a color coaugmented coalgebra C colored by X is defined by:

$$F_0^Q C = C[X], \qquad F_r^Q C = \left\{ x \in C \mid \overline{\Delta} \in F_{r-1}^Q C \boxtimes F_{r-1}^Q C \right\}. \tag{2.4}$$

If this filtration is exhaustive $C = \bigcup_{i\geqslant 0} F_i^Q C$, then the coalgebra will be called *color connected*.

If k is a field and C is colored by e, this agrees with original definition of [56], thus in the case of just one color, we will say that C is *Quillen connected* precisely if it is color connected. A graded coalgebra is Quillen connected if the degree zero part is isomorphic to R; in the literature R is often assumed to be a field k.

Theorem 2.18. *Let C be a colored coaugmented coalgebra colored by X. Then, for every $c \in \overline{C}_{x,y}$, $\Delta(c)$ can be written as $x \otimes c + c \otimes y + w$ with $w \in \overline{C} \boxtimes \overline{C}$, that is $w = \overline{\Delta}(X)$.*

Additionally assuming that $C = \bigoplus_{i\geqslant 0} C_i$ is graded and color reduced, and thus also $\overline{C}$ is graded, we moreover have that for $c \in \overline{C}_n$ and $n > 0$, $w \in \bigoplus_{i=1}^{n-1} C_i \boxtimes C_{n-i}$.

Proof. First, say that $c \in \overline{C}_{x,y}$ with $x \neq y$. We decompose (omitting the additional Sweedler sum symbols)

$$\Delta(c) = \sum_t c_{x,t}^{(1)} \otimes c_{t,y}^{(2)} = \left(r^{(1)}x + \bar{c}_{x,x}^{(1)}\right) \otimes \bar{c}_{x,y}^{(2)} + \bar{c}_{x,y}^{(1)} \otimes \left(r^{(2)}y + \bar{c}_{y,y}^{(2)}\right) + \sum_{z \notin \{x,y\}} \bar{c}_{x,z}^{(1)} \otimes \bar{c}_{z,y}^{(2)},$$

where we used the decomposition $C_{x,x} = Rx \oplus \overline{C}_{xx}$ and $C_{x,y} = \overline{C}_{x,y}$ for $x \neq y$. Applying the left counit constraint $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$ to c, we see that $r^{(1)} \bar{c}_{x,y}^{(2)} = c$ and the right unit constraint gives $\bar{c}_{x,y}^{(1)} r^{(2)} = c$. For $c \in \overline{C}_{x,x}$

$$\Delta(c) = \sum_t c_{x,t}^{(1)} \otimes c_{t,x}^{(2)} = \left(r^{(1)}x + \bar{c}_{x,x}^{(1)}\right) \otimes \left(r^{(2)}x + \bar{c}_{x,x}^{(2)}\right) + \sum_{z \neq x} \bar{c}_{x,z}^{(1)} \otimes \bar{c}_{z,y}^{(2)}.$$

From the left unit constraint, we obtain that $c = r^{(1)}r^{(2)}x + r^{(1)}\bar{c}_{x,x}^{(2)}$, but as $\epsilon(c) = 0$, $r^{(1)}r^{(2)} = 0$ and $r^{(1)}\bar{c}_{x,x}^{(2)} = c$. Similarly from the right unit constraint $\bar{c}_{x,x}^{(1)} r^{(2)} = c$.

In the graded case $\Delta(C_n) \subset \bigoplus_{i=0}^n C_i \boxtimes C_{n-i}$ and $w \in \bigoplus_{i=1}^n \overline{C}_i \boxtimes \overline{C}_{n-i} \subset \bigoplus_{i=0}^n C_i \boxtimes C_{n-i}$, but $\overline{C}_0 = 0$, so the first and the last summand vanish. $\blacksquare$

Lemma 2.19. *In a color coaugmented coalgebra colored by X the reduced comultiplication is coassociative, and thus there are unique n-th iterates $\overline{\Delta}^{[n]} : C \to C^{\otimes n}$.*

Proof. Because of linearity, it suffices to show it holds for $c \in C_{x,y}$. Using the following abbreviated Sweedler notation for the comultiplication $\Delta(c) = c^{(1)} \otimes c^{(2)}$ and $\Delta^{(3)}(c) = c^{(1)} \otimes c^{(2)} \otimes c^{(3)}$, calculating the left and right hand sides yields:

$$\begin{aligned}
\left(\bar{\Delta} \otimes \mathrm{id}_C\right) \circ \bar{\Delta}(c) &= \left(\bar{\Delta} \otimes \mathrm{id}_C\right) \circ \bar{\Delta}(c) = \left(\bar{\Delta} \otimes \mathrm{id}_C\right)\left(\Delta - \tilde{\lambda} - \tilde{\rho}\right)(c) \\
&= \left(\left(\Delta - \tilde{\lambda} - \tilde{\rho}\right) \otimes \mathrm{id}_C\right)\left(\Delta(c) - x \otimes c - c \otimes y\right) \\
&= \left(\Delta - \tilde{\lambda} - \tilde{\rho}\right)\left(c^{(1)}\right) \otimes c^{(2)} - \left(\Delta(x) - \tilde{\lambda}(x) - \rho(x)\right) \otimes c \\
&\quad - \left(\Delta(c) - \tilde{\lambda}(c) - \tilde{\rho}(c)\right) \otimes y \\
&= \Delta\left(c^{(1)}\right) \otimes c^{(2)} - x \otimes c^{(1)} \otimes c^{(2)} - \tilde{\rho}\left(c^{(1)}\right) \otimes c^{(2)} \\
&\quad - (x \otimes x - x \otimes x - x \otimes x) \otimes c - \left(c^{(1)} \otimes c^{(2)} - x \otimes c - c \otimes y\right) \otimes y \\
&= c^{(1)} \otimes c^{(2)} \otimes c^{(3)} - x \otimes c^{(1)} \otimes c^{(2)} - \tilde{\rho}\left(c^{(1)}\right) \otimes c^{(2)} \\
&\quad + x \otimes x \otimes c - c^{(1)} \otimes c^{(2)} \otimes y + x \otimes c \otimes y + c \otimes y \otimes y, \\
\left(\mathrm{id}_C \otimes \bar{\Delta}\right) \circ \bar{\Delta}(c) &= \left(\mathrm{id}_C \otimes \bar{\Delta}\right) \circ \bar{\Delta}(c) = \left(\mathrm{id}_C \otimes \bar{\Delta}\right)\left(\Delta - \tilde{\lambda} - \tilde{\rho}\right)(c) \\
&= \left(\mathrm{id}_C \otimes \left(\Delta - \tilde{\lambda} - \tilde{\rho}\right)\right)\left(\Delta(c) - x \otimes c - c \otimes y\right) \\
&= c^{(1)} \otimes \left(\Delta - \tilde{\lambda} - \tilde{\rho}\right)\left(c^{(2)}\right) - x \otimes \left(\Delta(c) - \tilde{\lambda}(c) - \tilde{\rho}(c)\right) \\
&\quad - c \otimes \left(\Delta(y) - \tilde{\lambda}(y) - \tilde{\rho}(y)\right) \\
&= \left(c^{(1)} \otimes c^{(2)} \otimes c^{(3)} - c^{(1)} \otimes \tilde{\lambda}\left(c^{(2)}\right) - c^{(1)} \otimes c^{(2)} \otimes y\right) \\
&\quad - x \otimes \left(c^{(1)} \otimes c^{(2)} - x \otimes c - c \otimes y\right) - c \otimes (y \otimes y - y \otimes y - y \otimes y) \\
&= c^{(1)} \otimes c^{(2)} \otimes c^{(3)} - c^{(1)} \otimes \tilde{\lambda}\left(c^{(2)}\right) - c^{(1)} \otimes c^{(2)} \otimes y - x \otimes c^{(1)} \otimes c^{(2)} \\
&\quad + x \otimes x \otimes c + x \otimes c \otimes y + c \otimes y \otimes y.
\end{aligned}$$

These agree if $\tilde{\rho}\left(c^{(1)}\right) \otimes c^{(2)} = c^{(1)} \otimes \tilde{\lambda}\left(c^{(2)}\right)$, which readily follows the assumption of being color coaugmented. $\blacksquare$

The lemma allows us to introduce the colored conilpotent coradical filtration by

$$F_0^N C = i(C[X]), \qquad F_r^N C = i(C[X]) \oplus \big\{ x \in C \,|\, \overline{\Delta}^{[r+1]}(x) = 0 \big\}, \qquad r \geqslant 1. \tag{2.5}$$

This is the generalization of the conilpotent coradical filtration, see, e.g., [48].

Definition 2.20. We call a color coaugmented co-algebra colored by $C[X]$ *conilpotent* if the filtration $F^N C$ is exhaustive. A color coaugmented coalgebra colored by X is called $\overline{\Delta}$ *flat*, if for all n, $\ker\big(\overline{\Delta}^{[n]} \otimes \mathrm{id}\big) = \ker\big(\overline{\Delta}^{[n]}\big) \otimes C$ and and $\ker\big(\mathrm{id} \otimes \overline{\Delta}^{[n]}\big) = C \otimes \ker\big(\overline{\Delta}^{[n]}\big)$.

For a coaugmented coalgebra with one color e, the filtration coincides with the *conilpotent coradical filtration* and color conilpotence agrees with notion of conilpotence for coaugmented coalgebras, see, e.g., [48].

Corollary 2.21. *A graded color coaugmented color reduced coalgebra colored by X is conilpotent.*

Proof. This follows directly from Theorem 2.18. ■

The notions of color conilpotent and color connected are related. Color conilpotence is easier to check, but color connectedness is better to argue with.

Proposition 2.22. *A color connected coalgebra is a color conilpotent coalgebra and a $\overline{\Delta}$-flat conilpotent coalgebra is color connected.*

Proof. If C is color connected, say $c \in F_n^Q(C) \cap \overline{C}$, by induction there is some N such that $\overline{\Delta}^{[N]}(c) = 0$. This is clear for $c \in F_0^Q C$ and if $c \in F_{n+1}^Q C \cap \overline{C}$ then $\overline{\Delta}(c) \in F_n^Q C \otimes F_n^Q C$ so there is some N such that $\big(\overline{\Delta}^{[N]} \otimes \mathrm{id}\big)\overline{\Delta}(c) = \overline{\Delta}^{[N+1]}(c) = 0$. Vice-versa, assume that C is $\overline{\Delta}$-flat color conilpotent. Again using induction, as $F_0^N C = F_0^Q C$, we can assume that $F_{n-1}^N C \subset F_{n-1}^Q C$. Now, if $c \in \ker\big(\overline{\Delta}^{[n]}\big)$, then, since C is $\overline{\Delta}$-flat, $\overline{\Delta}(c) \in \ker\big(\overline{\Delta}^{[n-1]} \otimes \mathrm{id}\big) \cap \ker\big(\mathrm{id} \otimes \overline{\Delta}^{[n-1]}\big) = \ker\big(\overline{\Delta}^{[n-1]}\big) \otimes C \cap C \otimes \ker\big(\overline{\Delta}^{[n-1]}\big) \subset F_{n-1}^Q F \otimes F_{n-1}^Q C$ and $c \in F_n^Q C$. ■

Lemma 2.23. *A color coaugmented coalgebra C colored by X is $\overline{\Delta}$-flat if C is flat. In particular, $\overline{\Delta}$ flatness is automatic over a field or if C is free.*

Proof. Consider the short exact sequence $0 \to \ker\big(\overline{\Delta}^{[n]}\big) \to C \to \mathrm{im}\big(\overline{\Delta}^{[n]}\big) \to 0$. By right exactness of $\otimes$: $\ker\big(\overline{\Delta}^{[n]}\big) \otimes C \to C \otimes C \to \mathrm{im}\big(\overline{\Delta}^{[n]}\big) \otimes C \to 0$ is exact. If C is flat, $0 \to \mathrm{im}\big(\overline{\Delta}^{[n]}\big) \otimes C^{\otimes n} \to C \otimes C$ is exact and hence $\ker\big(\overline{\Delta}^{[n]}\big) \otimes C \overset{i \otimes \mathrm{id}}{\to} C \otimes C \overset{\overline{\Delta}^{[n]} \otimes \mathrm{id}}{\to} C^{\otimes n} \otimes C$ is exact and $\ker\big(\overline{\Delta}^{[n]} \otimes \mathrm{id}\big) = \ker\big(\overline{\Delta}^{[n]}\big) \otimes C$. ■

Proposition 2.24. *For a decomposition finite colored monoid M, the categorical coalgebra $C[M]$ is colored by its objects and color coaugmented.*

If M only has identities as invertibles, then $C[M]$ is colored by its grouplikes. If additionally, $C[M]$ is color nilpotent, then it is color connected.

In particular, if M has a proper degree function, is locally finite and has no non-trivial identities, then it is colored by its grouplikes and is color connected.

Proof. The first statement is clear by construction. If identities are the only invertible elements, then by Lemma 2.12 they are grouplike, and these are also the only semigrouplike elements. Hence, $C[M]$ is colored by its grouplikes. As $C[M]$ is free being color nilpotent implies being color connected by Lemma 2.23 and Proposition 2.22. The last statement follows from the previous ones and Corollary 2.21. ■

2.3.2 Colored bialgebras and the free bialgebra $FB[M]$ on a colored monoid

Generally, we will say that a bialgebra has an attribute like colored, color connected etc. if its coalgebra does.

Lemma 2.25. *Given a colored connected bialgebra colored by X the ideal I generated by $x - y$, for $x, y \in X$ is a coideal and C/I is Quillen connected.*

Proof. Indeed, $\epsilon(x - y) = 1 - 1 = 0$ and $\Delta(x - y) = x \otimes x - y \otimes y = x \otimes (x - y) + (x - y) \otimes y$, so I is a coideal. C/I is single colored and coaugmented as $C[X]/I = R$. The induced filtration $F_r^Q(C/I) = F_r^Q C / (I \cap F_r^Q C)$ is still exhaustive. ∎

The following construction is key for many examples. In particular for constructing bialgebras from posets or quivers.

For a set X let $X^\times = \amalg_{n \in \mathbb{N}_0} X^{\times n}$ be the free monoid on X. The unit 1 is represented by the empty product. If M is a monoid then $M^\times$ additionally carries a componentwise monoidal structure: $(a_1, \dots, a_n)(b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n)$. Consider $R^{(M^\times)}$ as the *free unital algebra* on $M^\times$. If M is decomposition finite, this has a coalgebra structure induced by that of $C[M]$, whose comultiplication is given by $\Delta(1) = 1 \otimes 1$ and $\Delta((a_1, \dots, a_n)) = \sum (a_1^{(1)}, \dots, a_n^{(1)}) \otimes (a_1^{(2)}, \dots, a_n^{(2)})$, where $\Delta(a_i) = \sum a_i^{(1)} \otimes a_i^{(2)}$ is the coproduct of $C[M]$. The counit is $\epsilon((a_1, \dots, a_n)) = \epsilon(a_1) \cdots \epsilon(a_n)$. These structures make $R^{(M^\times)}$ into a unital counital bialgebra with counit ϵ and unit 1, which we will denote by $FB[M]$.

Proposition 2.26. *If M is a monoid colored by X, then $M^\times$ is colored by $X^\times$. In this case the underlying algebra structure of $FB[M]$ is graded by the monoid $X^\times$ and the underlying coalgebra of $FB[M]$ is a $X^\times$ colored, and Proposition 2.24 applies.*

Proof. The coloring is given by $s(a_1, \dots, a_n) = (s(a_1), \dots, s(a_n))$ and $t(a_1, \dots, a_n) = (t(a_1), \dots, t(a_n))$. The computations are then straightforward. ∎

In this construction the original composition is turned into a coproduct. Then a free product is added to make a bialgebra. If M is interpreted as a category $\mathcal{C}$, this is the categorical coalgebra for the free monoidal category $\mathcal{C}^\otimes$. The case were there already is a monoidal structure, which is not necessarily free, is treated in Section 5.

3 Path-like co- and bialgebras, convolution inverses and antipodes

3.1 QT-filtration and convolution inverses

In order to define antipodes, or more generally convolution inverses, it is useful to regard filtrations, as in good cases one can recursively build up the antipode from its value on the degree 0 part. To this end, we put forward the definition of a QT-filtration (Quillen–Takeuchi) that generalizes the classical graded, coalgebra, coradical, Quillen [56] filtrations and their colored conilpotent versions. This notion is what is needed to prove a general version of Takeuchi's Lemma [60], for the existence of convolution inverses.

Definition 3.1. A *QT-filtration* of a coalgebra C is a filtration of C by R-submodules $F_p^{QT}(C)$, $p \geqslant 0$ such that:

1. $F_0^{QT} C = S$ is a subcoalgebra.
2. $\Delta(F_p^{QT} C) \subset F_{p-1}^{QT} C \boxtimes C + C \boxtimes F_{p-1}^{QT} C.$

We call a QT-filtration a *QT-sequence* if the filtration is exhaustive, i.e., $C = \bigcup_{i=0}^{\infty} F_i C$ and call the QT-sequence *split*, if $F_0^{QT}C$ is a direct summand, that is $0 \to F_0^{QT}C \to C \to F_0^{QT}C/C \to 0$ is split.

A coaugmented coalgebra is *QT-connected if is has a QT-sequence whose degree 0 part is* Re.

Remark 3.2. Note that a QT-filtration is defined by specifying a base, that is a subcoalgebra S and setting: $F_0^{QT}C = S$ and $F_r^{QT}C = \left\{ x \mid \Delta(x) \in C \boxtimes F_0^{QT}C + F_{p-1}^{QT}C \boxtimes F_{p-1}^{QT}C + F_0^{QT}C \boxtimes C \right\}$. Quillen's original filtration [56] for a coaugmented coalgebra is obtained from $S = F_0^{QT} = $ Re. In the category of R-coalgebras, every coalgebra filtration gives rise to a QT-filtration with S being the degree 0 part.

The filtration associated with the grading of an $\mathbb{N}$-graded coalgebra $\left(F_r^{QT} := \{x \mid \deg x \leqslant r\} \right)$ is a QT-filtration based on F_0^{QT}, because $F_i^{QT} \subset F_r^{QT}$ if $i \leqslant r$. $F_i^{QT} \boxtimes F_j^{QT} \subset F_r^{QT} \boxtimes F_q^{QT}$ if $i \leqslant r$ and $j \leqslant q$. Given a $\mathbb{Z}$ graded filtration, with C_0 a subcoalgebra, and with $\Delta(F_p) \subset F_{p-1} \boxtimes C + C \boxtimes F_{p-1}$, $p > 0$ and $\Delta(F_p) \subset F_{p+1} \boxtimes C + C \boxtimes F_{p+1}$, $p < 0$, setting $F_p^{QT} = F_p + F_{-p}$, $p \geqslant 0$ produces a QT-filtration.

Theorem 3.3. *Consider a coalgebra C with counit ϵ and an algebra A with unit η. Given a QT-sequence $\left\{ F_i^{QT} \right\}_{i=0}^{\infty}$ on C, such that $\operatorname{Ext}_R^1\left(C/F_0^{QT}C, A \right) = 0$, for any element f of $\operatorname{Hom}(C, A)$, f is $\star$-invertible if and only if the restriction $f|_{F_0^{QT}C}$ is $\star$-invertible in $\operatorname{Hom}\left(F_0^{QT}C, A \right)$.*

Proof. First assume that f is convolution invertible in $\operatorname{Hom}(C, A)$, and let g be $\star$-inverse of f. By restriction to $F_0^{QT}C$, see Lemma 2.2, it follows that $(\eta \circ \epsilon)|_{F_0^{QT}C} = (f \star g)|_{F_0^{QT}C} = f|_{F_0^{QT}C} \star g|_{F_0^{QT}C}$ and $(\eta \circ \epsilon)|_{F_0^{QT}C}$ is the unit of the restricted convolution algebra $\operatorname{Hom}\left(F_0^{QT}C, A \right)$. Thus $g|_{F_0^{QT}C}$ is a right $\star$-inverse of $f|_{F_0^{QT}}$. By symmetry, it is also a left $\star$-inverse and thus the $\star$-inverse in the restricted algebra.

Conversely, assume that $f|_{F_0^{QT}C}$ is convolution invertible in $\operatorname{Hom}\left(F_0^{QT}C, A \right)$. Thus, there is an element g of $\operatorname{Hom}\left(F_0^{QT}C, A \right)$ such that $f \star g = \eta \circ \epsilon = g \star f$ on $F_0^{QT}C$. We can extend the R-linear map g to C as the restriction morphisms $\operatorname{Hom}(C, A) \to \operatorname{Hom}\left(F_0^{QT}C, A \right)$ is surjective due to the assumption that $\operatorname{Ext}\left(C/F_0^{QT}C, A \right) = 0$. Because the convolution algebra $(\operatorname{Hom}(C, A), \star, \eta \circ \epsilon)$ is a unital monoid, if we show that both $f \star g$ and $g \star f$ have a $\star$-inverse this will imply f has a $\star$-inverse. Thus, replacing f with $f \star g$, we may assume $f|_{F_0^{QT}C} = \eta \circ \epsilon|_{F_0^{QT}C}$ that is $\eta \circ \epsilon|_{F_0^{QT}C} - f|_{F_0^{QT}C} = 0$. Since the filtration is exhaustive, for every $x \in C$, there exists a natural number N such that $x \in F_N$ and $(f - \eta \circ \epsilon)^{\star(N)}(x) = 0$. Indeed, after using the coassociativity for the diagonal to decrease the filtration degree, every summand from $\Delta^{[N+1]}(x) = \sum_x x_{(1)} x_{(2)} x_{(3)} \cdots x_{(N+1)}$ has to contain a factor in F_0C according to the definition of the QT-filtration, and hence $\eta \circ \epsilon - f$ annihilates each term. So, the infinite sum $h = \sum_{i=0}^{\infty} (\eta \circ \epsilon - f)^{\star i}$ is locally finite and yields a well defined map, where we set $(\eta \circ \epsilon - f)^{\star 0} = \eta \circ \epsilon$. It is easy to verify that h is R-linear, so $h \in \operatorname{Hom}(C, A)$. That h is the inverse of f follows by computation: $f \star h = (f - \eta \circ \epsilon) \star h + \eta \circ \epsilon \star h = (f - \eta \circ \epsilon) \star h + h = -\sum_{i=1}^{\infty} (\eta \circ \epsilon - f)^{\star i} + \sum_{i=0}^{\infty} (\eta \circ \epsilon - f)^{\star i} = (\eta \circ \epsilon - f)^{\star 0} = \eta \circ \epsilon$. The fact that $h \star f = \eta \circ \epsilon$ can be checked in the same way. Therefore, we conclude h is $\star$-inverse of f. ∎

Note that if $F_0^{QT}C$ is a direct summand, i.e., the sequence $0 \to F_0^{QT}C \to C \to C/F_0^{QT}C \to 0$ splits, the Ext group vanishes, and the first condition is automatically met.

We say that a bialgebra has a QT-sequence, if the coalgebra has a QT-sequence. In this situation, Theorem 3.3 yields the generalization of the results of Quillen and Takeuchi, and we recover many of the existence theorems for antipodes as special cases.

Theorem 3.4. *A bialgebra with a QT-sequence and $\operatorname{Ext}_R^1\left(C/F_0^{Q}C, C \right) = 0$ is Hopf, i.e., has an antipode, if and only if $\operatorname{id}|_{F_0^{QT}C}$ has a $\star$ inverse.*

Corollary 3.5. *A bialgebra with a split QT-sequence is Hopf if and only if* $\mathrm{id}_{F_0^Q C}$ *has a convolution inverse. In particular, if C is a split QT-connected bialgebra, then it is Hopf.*

Proof. The first part directly follows from Theorem 3.4. If C is QT-connected then $S(e) = e$ is the required antipode on $F_0^Q C$. ∎

Proposition 3.6. *The Quillen filtration $F^Q C$, cf. (2.4), for a color coaugmented coalgebra colored by X is a QT-filtration and a QT-sequence if C is connected. For a $\overline{\Delta}$-flat color coaugmented coalgebra colored by X the conilpotent filtration $F^N C$, cf. (2.5), is a QT-filtration with $F_0^{QT} C = C[X]$ which is exhaustive if $\overline{\Delta}$ is conilpotent.*

Proof. The first part follows from the fact that in this situation, $\overline{\Delta} = \bigoplus_{g,h \in X \times X} \Delta_{g,h}$ and thus $F_r^{QT} C \supset F_r^Q(C)$ with $F_0^{QT} C = C[X] = \bigoplus_{g \in \mathrm{sgpl}(C)} Rg$. For the second part: decompose $c \in F_r(C)$ as $c_0 + \bar{c} \in C = C[X] \oplus \bar{C}$ with $\bar{c}$ in $\ker\big(\overline{\Delta}^{[r]}\big)$, then by Theorem 2.18 $\Delta(c) = c_0 \otimes c_0 + i_X \lambda(\bar{c}) \otimes \bar{c} + c \otimes i_X \rho(\bar{c}) + \overline{\Delta}(\bar{c})$ with $\overline{\Delta}(\bar{c}) \in \ker\big(\overline{\Delta}^{[r-1]}\big) \otimes \mathrm{id}$. Thus, by definition of $\overline{\Delta}$-flatness $\Delta(c) \in F_0 C \boxtimes C \oplus C \boxtimes F_0 C \oplus F_{r-1} \boxtimes C$. ∎

Corollary 3.7. *A bialgebra whose coalgebra is colored by X and color connected, or whose coalgebra is a $\overline{\Delta}$-flat color coaugmented coalgebra with colors X has an antipode if and only if every generator $x \in C[X]$ has a algebra inverse x^{-1}.*

The Proposition allows to give a recursive definition in both cases.

3.2 Sg-flavored bi- and coalgebras

In this section, we introduce a type of coalgebra which has a QT-filtration based on a decomposition according to (g, h)-primitives. This generalizes path-coalgebras and colored coalgebras and subsumes the notions of graded connected coalgebras and May's component coalgebras [51]. This type of idea is also used in [59].

Definition 3.8. Let C be a R-coalgebra and g and h be semigrouplike elements. We define the (g, h)-reduced comultiplication to be $\overline{\Delta}_{g,h}(x) = \Delta(x) - x \otimes g - h \otimes x$.

Lemma 3.9. *Pairs of $\Delta_{g,h}s$ have the following associators:*

$$\big(\overline{\Delta}_{g,h} \otimes \mathrm{id}\big)\overline{\Delta}_{g,h}(x) - \big(\mathrm{id} \otimes \overline{\Delta}_{g,h}\big)\overline{\Delta}_{g,h}(x)$$
$$= x^{(1)} \otimes (h - g) \otimes x^{(2)} - x \otimes (h - g) \otimes g - h \otimes (h - g) \otimes x.$$

The $\overline{\Delta}_g := \Delta_{g,g}$ are coassociative, thus, the iterated reduced comultiplication $\overline{\Delta}_g^{[n]}(x)$ is well-defined. Furthermore, if g is grouplike and $x \in \ker \epsilon$, then $(\epsilon \otimes \mathrm{id})\overline{\Delta}_g(x) = (\mathrm{id} \otimes \epsilon)\overline{\Delta}_g(x) = 0$.

Proof. The first claim follows from the calculation:

$$\big(\overline{\Delta}_{g,h} \otimes \mathrm{id}\big)\overline{\Delta}_{g,h}(x) - \big(\mathrm{id} \otimes \overline{\Delta}_{g,h}\big)\overline{\Delta}_{g,h}(x)$$
$$= \sum_x x^{(11)} \otimes x^{(12)} \otimes x^{(2)} - \sum_x x^{(11)} \otimes x^{(21)} \otimes x^{(22)} - x^{(1)} \otimes g \otimes x^{(2)}$$
$$- h \otimes x^{(1)} \otimes x^{(2)} - x^{(1)} \otimes x^{(2)} \otimes g + x \otimes g \otimes g + h \otimes x \otimes g + h \otimes g \otimes x$$
$$- \big[x^{(1)} \otimes x^{(2)} \otimes g - x^{(1)} \otimes h \otimes x^{(2)} + x \otimes h \otimes g - h \otimes x^{(1)} \otimes x^{(2)} + h \otimes x \otimes g$$
$$+ h \otimes h \otimes x\big] = x^{(1)} \otimes (h - g) \otimes x^{(2)} - x \otimes (h - g) \otimes g - h \otimes (h - g) \otimes x.$$

The second claim is straightforward, e.g., $(\epsilon \otimes \mathrm{id})\overline{\Delta}_g(x) = x - \epsilon(x)g - \epsilon(g)x = 0$. ∎

Given a coalgebra C, we will inductively define R-modules each depending on two semigroup-like elements $g, h \in \mathrm{sgpl}(C)$. These are simultaneously defined for all pairs of such elements and yield pieces of a filtration.

1. For any two $g, h \in \mathrm{sgpl}(C)$: $F_0^Q C_{g,h} := Rg \cap Rh$ is the base R-module.

2. Inductively over all pairs: $F_n^Q C_{g,h} := \big\{ x \,|\, \overline{\Delta}_{g,h}(x) = \in \sum_{t \in \mathrm{sgpl}(C)} F_{n-1}^Q C_{g,t} \boxtimes F_{n-1}^Q C_{t,h} \big\}$.

In words, $F_n^Q C_{g,h}$ is the subset in which every element x has the property that if $\overline{\Delta}_{g,h}(x) = \sum x^{(1)} \otimes x^{(2)}$ then for each summand $x^{(1)} \otimes x^{(2)}$ there exists a semigrouplike element t with $x^{(1)} \in F_{n-1}^Q C_{g,t}$ and $x^{(2)} \in F_{n-1}^Q C_{t,h}$. This is also a R-module as every $F_n^Q C_{g_1, g_2}$ with $g_1, g_2 \in \mathrm{sgpl}(C)$ is a R-module inductively. The (g,h)-*Quillen component* of C is defined to be $F^Q C_{g,h} = \sum_{i \geqslant 0} F_i^Q C_{g,h}$.

Lemma 3.10. *The $F_i^Q C_{g,h}$ yield filtrations of the $F^Q C_{g,h}$; that is $F_i C_{g,h}^Q \subset F_{i+1} C_{g,h}^Q$ for $i \geqslant 0$.*

Proof. We use induction. Base case: to show that $Rg \cap Rh = F_0^Q C_{g,h} \subset F_1^Q C_{g,h}$ pick $x \in F_0^Q C_{g,h}$, say $x = rg$, then $\overline{\Delta}_{g,h}(x) = -x \otimes h \in F_0 C_{g,h}^Q \boxtimes F_0 C_{h,h}^Q$. By induction assume that $F_{n-1} C_{g,h}^Q \subset F_n C_{g,h}^Q$. Let $x \in F_n^Q C_{g,h}$. Then $\overline{\Delta}_{g,h}(x) \in \sum_{t \in \mathrm{sgpl}(C)} F_{n-1}^Q C_{g,t} \boxtimes F_{n-1} C_{t,h}^Q \subset \sum_{t \in \mathrm{sgpl}(C)} F_n C_{g,t}^Q \boxtimes F_n C_{t,h}^Q$ by induction hypothesis, and thus $x \in F_{n+1} C_{g,h}^Q$. ∎

Definition 3.11. The filtrations of $C_{g,h}^Q$ yield a common filtration $F_r^{Q \bullet\bullet} C := \sum_{g,h} F_r^Q C_{g,h}$ – the bivariate Quillen filtration. We say a coalgebra C is *sg-flavored* (sg stands for semigrouplike) if the bivariate Quillen filtration is exhaustive, that is $C = \sum_{(g,h), g, h \in \mathrm{sgpl}(C)} C_{g,h}^Q$ and *split* if $F_0^{Q \bullet\bullet} C$ is a direct summand. A bi- or Hopf-algebra is sg-flavored if the underlying coalgebra is sg-flavored.

Remark 3.12. Note all semigrouplike elements of C lie in $F_0^{Q \bullet\bullet} C$ and all skew primitive elements C lie in $F_1^{Q \bullet\bullet} C$. The degree zero component $F_0 C_{g,h}^Q$ may be non-empty, if g, h are two distinct semigrouplike elements, it may happen that there exist $s, t \in R$ such that $sg = th$ so that $Rg \cap Rh \neq 0$. This complication does not appear over a ground field $R = k$ or if the respective submodules are free over $\mathbb{Z}$, as the equation then implies that $g = h$.

$C_{g,h}^Q$ is never empty. This is clear for $g = h$. For $g \neq h$ $\Delta(g - h) = g \otimes (g - h) + (g - h) \otimes h$ and hence $\overline{\Delta}_{g,h}(g - h) = 0 \in F_0^{Q \bullet\bullet} C_{g,t} \boxtimes F_0^{Q \bullet\bullet} C_{t,h}$ for any t. Thus, if there is more than one semigrouplike element, the sum $C = \sum_{g,h} C_{g,h}^Q$ is never direct, as $g - h$ is in $\big(C_{g,g}^Q + C_{h,h}^Q \big) \cap C_{g,h}^Q$.

In the case of a field $R = k$, $g - h$ generates the intersection and in fact over a field one can make the sum direct by reducing $C_{g,h}$ by $k(g - h)$ following the line of arguments presented in [53]. Note that the corresponding splittings are not unique.

Lemma 3.13. *The R-modules $F_r^{Q \bullet\bullet} C := \sum_{g,h} F_r C_{g,h}^Q$ for $r \geqslant 0$, form a QT-filtration, which is a QT-sequence if C is sg-flavored.*

Proof. By definition the elements in $F_0^{Q \bullet\bullet} C$ are multiples of semigrouplike elements and hence $F_0 C$ is a subcoalgebra. Condition (2) for a QT-filtration is satisfied by construction. In the sg-flavored case, the filtration is exhaustive by definition. ∎

Theorem 3.14. *For an sg-flavored coalgebra with $\mathrm{Ext}_R^1\big(C / F_0^{Q \bullet\bullet} C, A \big) = 0$, an element f of $\mathrm{Hom}(C, A)$, f is $\star$-invertible if and only if the restriction $f|_{F_0^{QT} C}$ is $\star$-invertible in $\mathrm{Hom}\big(F_0^{Q \bullet\bullet} C, A \big)$.*

In particular, a split sg-flavored bialgebra is a Hopf algebra if $F_0^{Q \bullet\bullet} C$ is the quotient of a group algebra by a Hopf ideal.

Proof. By Lemma 3.13, Theorem 3.4 applies and we need that id has an antipode. Now on any semigrouplike there is only one possible value for the antipode: $S(g) = g^{-1}$, if it exists. Thus, S being defined on g is equivalent to g being invertible.

If g is grouplike, so is g^{-1}, so that $g^{-1} \in F_0^{Q\bullet\bullet}C$. This follows from applying the counit constraints to $1 \otimes 1 = \Delta(1) = \Delta(gg^{-1}) = \sum g(g^{-1})^{(1)} \otimes g(g^{-1})^{(2)}$. Thus for the second statement, the antipode is fixed on the generators and descends precisely if the quotient is by a Hopf ideal. ∎

The following proposition explains the usual constructions of antipodes which pass through a connected quotient.

Proposition 3.15. *Let B be a sg-flavored split bialgebra all of whose semigrouplikes are grouplike, then the ideal I spanned by $g - h$ for $g, h \in \mathrm{gpl}(C)$ is a coideal and $B^{\mathrm{red}} = B/I$ is a Hopf algebra.*

Proof. As before $\epsilon(g - h) = 1 - 1 = 0$ and $\Delta(g - h) = g \otimes (g - h) + (g - h) \otimes h$. This filtration is exhaustive, as it is induced by the quotient $F_r^{Q\bullet\bullet}(C/I) = F_r^{Q\bullet\bullet}(C)/(I \cap F_r^{Q\bullet\bullet})$. The bivariate Quillen filtration on the quotient is simply the Quillen filtration. The splitting passes to the quotient and as a Quillen connected bialgebra C/I is Hopf. ∎

It is possible to truncate to only the grouplike and skew primitive elements.

Proposition 3.16. *If a bialgebra B is generated by grouplike and skew primitive elements, then it is sg-flavored.*

Proof. Observe that if g, h are grouplike elements and x a skew primitive element, then gh is grouplike and gx is a skew primitive. Therefore, we focus on the products of skew primitives $x_n \cdots x_1$. If x is a (g, h)-skew primitive and x' is a (g', h')-skew primitive, it follows that

$$\Delta(xx') - (gg' \otimes xx' + xx' \otimes hh') = xg' \otimes hx' + gx' \otimes xh',$$

where xg' is a (gg', hg')-skew primitive, hx' is a (hg', hh')-skew primitive, gx' is a (gg', gh')-skew primitive and xh' is a (gh', hh')-skew primitive. Therefore,

$$\begin{aligned}
\Delta(xx') - (gg' \otimes xx' + xx' \otimes hh') &\in F_1^{Q\bullet\bullet}C_{gg',hg'} \boxtimes F_1^{Q\bullet\bullet}C_{hg',hh'} \\
&+ F_1^{Q\bullet\bullet}C_{gg',gh'} \boxtimes F_1^{Q\bullet\bullet}C_{gh',hh'}
\end{aligned}$$

and $xx' \in F_2^{Q\bullet\bullet}C_{gg',hh'}$. We claim any word of skew primitive elements $x_n \cdots x_1$ belongs to a component. We proceed by induction with the above observation providing the base case. Suppose $x_1 \cdots x_{n-1}$ belongs to some component for every possible product of skew primitive elements. Applying Lemma 3.17 to $x_n(x_{n-1} \cdots x_1)$ finishes that proof. ∎

Lemma 3.17. *Let B be a bialgebra and g, h, p, q be grouplikes and x be (p, q)-skew primitive, then $pC_{g,h}^Q \subset C_{pg,ph}^Q$ and $xC_{g,h}^Q \subset C_{pg,qh}^Q$.*

Proof. In fact, $pF_nB_{g,h}^Q \subset F_nB_{pg,ph}^Q$ and $xF_nB_{g,h}^Q \subset F_{n+1}C_{pg,qh}^Q$. We prove these relations by induction. For the base case, we use the fact that the multiplication of two grouplike elements is grouplike and the multiplication of a grouplike element and a skew primitive element is skew primitive. In particular, we have gx is a (gp, gq)-skew primitive element. Now we assume $pF_{n-1}B_{g,h}^Q \subset F_{n-1}B_{pg,ph}^Q$ Observe for $z \in F_nB_{g,h}^Q$ and $\overline{\Delta}_{g,h}(z) = \sum z_{(1)} \otimes z_{(2)} \in \sum_{t \in \mathrm{sgpl}(C)} F_{n-1}B_{g,t}^Q \boxtimes F_{n-1}B_{t,h}^Q$. By algebraic manipulations, it follows $\overline{\Delta}_{pg,ph}(pz) = \sum pz_{(1)} \otimes pz_{(2)}$, where $\overline{\Delta}_{pg,ph}(pz) \in \sum_{t \in \mathrm{sgpl}(C)} pF_{n-1}B_{g,t}^Q \boxtimes pF_{n-1}B_{t,h}^Q$. By induction hypothesis: $\overline{\Delta}_{pg,ph}(pz)$

$\in \sum_{t\in\mathrm{sgpl}(C)} F_{n-1}B^Q_{pg,pt} \boxtimes F_{n-1}B^Q_{pt,ph}$. Therefore, $pz \in F_nB^Q_{pg,ph}$ and $pF_nB^Q_{g,h} \subset F_nB^Q_{pg,ph}$. Now, we assume $xF_{n-1}B^Q_{g,h} \subset F_nC^Q_{pg,qh}$. Observe

$$\overline{\Delta}_{pg,qh}(xz) = xg \otimes qz + \sum pz_{(1)} \otimes xz_{(2)} + \sum xz_{(1)} \otimes qz_{(2)} + pz \otimes xh,$$

so

$$\overline{\Delta}_{pg,qh}(xz) \in F_1B^Q_{pg,qg} \boxtimes qF_nB^Q_{g,h} + \sum_{t\in\mathrm{sgpl}(C)} pF_{n-1}B^Q_{g,t} \boxtimes xF_{n-1}B^Q_{t,h}$$

$$+ \sum_{t\in\mathrm{sgpl}(C)} xF_{n-1}B^Q_{g,t} \boxtimes qF_{n-1}B^Q_{t,h} + pF_nB^Q_{g,h} \boxtimes F_1B^Q_{ph,qh}.$$

Using the previous result for grouplike elements and that $M \subset M'$ and $N \subset N'$ as submodules, then $M \boxtimes N \subset M' \boxtimes N'$.

$$\overline{\Delta}_{pg,qh}(xz) \in F_nB^Q_{pg,qg} \boxtimes F_nB^Q_{qg,qh} + \sum_{t\in\mathrm{sgpl}(C)} F_nB^Q_{pg,pt} \boxtimes F_nB^Q_{xt,xh}$$

$$+ \sum_{t\in\mathrm{sgpl}(C)} F_nB^Q_{xg,xt} \boxtimes F_nB^Q_{qt,qh} + F_nB^Q_{pg,ph} \boxtimes F_nB^Q_{ph,qh}.$$

Hence, $xz \in F_{n+1}B^Q_{pg,qh}$. ∎

Proposition 3.18. *Let B be a split sg-flavored bialgebra all of whose semigrouplikes are grouplike and let $\pi\colon B \to B^{\mathrm{red}}$ be the projection. The Brown map defined as*

$$\Delta_B\colon \quad B \to B \otimes_R B^{\mathrm{red}}, \qquad \Delta_B(x) = \sum x^{(1)} \otimes \pi\big(x^{(2)}\big)$$

is a Hopf coaction.

Proof. The result of both iterations is $\sum x^{(1)} \otimes \pi\big(x^{(2)}\big) \otimes \pi\big(x^{(3)}\big)$. ∎

3.3 Pathlike coalgebras

In many applications, we need a notion which is stronger than being sg-flavored, yet is weaker than being QT-connected.

Definition 3.19. A sg-flavored coalgebra is said to be *pathlike* if

1. $\mathrm{sgpl}(C) = \mathrm{gpl}(C)$.
2. $F_0^{Q^{\bullet\bullet}}C = R^{(\mathrm{gpl})} = \bigoplus_{g\in\mathrm{gpl}(C)} Rg$ as a submodule.

We say a pathlike coalgebra is split, if F_0C is a direct summand, i.e., the sequence of R modules $0 \to F_0C \to C \to C/F_0C \to 0$ is split.

A bi- or Hopf-algebra is pathlike or split pathlike if the underlying coalgebra is.

At the moment, we will disregard semigrouplike elements that are not grouplike. Handling such elements is more tricky and they only appear in torsion, but they might be of interest in the situation with modified counits, [25, Section 2.2] and for applications like [39].

Example 3.20. As expected by the name, the path coalgebra of a quiver Q is pathlike. Indeed, $\mathrm{sgpl} = \mathrm{gpl}$ equals the set of vertices $V(Q)$. The base is given by the sum of the $Rv \cap Rw = \delta_{v,w}R_v$, thus $F^{Q^{\bullet\bullet}}C = \bigoplus_{v\in V} Rv$. Note that $C_1C^Q_{g,h} = R\vec{e} + R(g-h)$, if there is a directed edge $\vec{e}$ from g to h. It is split and $F_0^{Q^{\bullet\bullet}}C$ is free.

The assumptions allow us to simplify and strengthen Theorem 3.14:

Theorem 3.21. *Let C be pathlike coalgebra and A algebra with $\mathrm{Ext}\left(C/F_0^{Q\bullet\bullet}C, A\right) = 0$. An element $f \in \mathrm{Hom}(C, A)$ has an $\star$-inverse in the convolution algebra $\mathrm{Hom}(C, A)$ if and only if for every grouplike element g, $f(g)$ has an inverse as ring element in A.*

In particular, in this situation, a character is $\star$-invertible if and only if it is grouplike invertible.

Furthermore, a pathlike bialgebra B is a Hopf algebra if and only if the set of grouplike elements form a group.

Proof. By Theorem 3.14, f having a $\star$-inverse is equivalent to the restriction of f on $\bigoplus_{g\in\mathrm{gpl}(C)} Rg$ considered as an element in $\mathrm{Hom}\left(\bigoplus_{g\in\mathrm{gpl}(C)} Rg, A\right)$ having a convolution inverse. Say that for each $g \in \mathrm{gpl}(C)$, $f(g) \in A$ has an inverse then setting $h(g) = f(g)^{-1}$ defines a convolution inverse on $F_0^{Q\bullet\bullet}C$, as $(f \star h)(g) = f(g)h(g) = 1_A = \eta_A(\epsilon_C(g))$ –symmetrically for $h \star g$. On the other hand, if f has a convolution inverse $f^{\star-1}$, then the extension of $h(g) = f(g)^{-1}$ as above also yields a convolution inverse and these must agree.

Using Corollary 3.5, and the theorem above, having an antipode on B is equivalent to having an antipode on $F_0^Q C = \bigoplus_{g\in\mathrm{gpl}} Rg$. The only possible antipode on grouplike elements is $S(g) = g^{-1}$ if the inverses exist. Hence the existence is equivalent to the existence of inverses. ∎

The following proposition will cover the targeted examples.

Proposition 3.22. *A color connected colored coalgebra C is a pathlike coalgebra. In particular, a $\overline{\Delta}$-flat color conilpotent colored coalgebra is a pathlike coalgebra.*

Proof. Let C be colored by $X = \mathrm{sgpl}(C)$. In view of Proposition 3.6, we need to check if the sg-flavored coalgebra is pathlike. Being color coaugmented means that all semigrouplike elements are grouplike elements, since $(1 - \epsilon(g))g = 0$, and the submodule Rg is free if and only if $\epsilon(g) = 1$. By definition $F_0^{Q\bullet\bullet}(C) = C[\mathrm{gpl}(C)] = \bigoplus_{g\in\mathrm{gpl}(C)} Rg$. ∎

Proposition 3.23. *The coopposite of an sg-flavored, respectively pathlike, coalgebra is also sg-flavored, respectively pathlike.*

Proof. First, $\mathrm{sgpl}(C^{\mathrm{cop}}) = \mathrm{sgpl}(C)$. Now, the definition of the filtration is symmetric and thus it is also exhaustive for C^{op}. The further conditions for pathlike then only concern the R-module structure, which remains unchanged. ∎

Remark 3.24. This means that there is a Drinfel'd double for sg-flavored or pathlike Hopf algebras, see, e.g., [32, Section IX.4].

Corollary 3.25. *The antipode of a split sg-flavored Hopf algebra H is bijective.*

Proof. Using Proposition 3.23 and Theorem 3.14, it follows that H^{cop} is also a Hopf algebra. Indeed, $\mathrm{id}|_{F_0^{Q\bullet\bullet}}$ is generated by $\mathrm{sgpl}(C)$, so that the antipode on the generators exists and is fixed as $S(g) = g^{-1}$. When restricted to $F_0^{Q\bullet\bullet}H^{\mathrm{cop}} = F_0^{Q\bullet\bullet}H$, S provides a convolution inverse for $\Delta^{\mathrm{op}}|_{F_0^{Q\bullet\bullet}H^{\mathrm{cop}}} = \Delta|_{F_0^{Q\bullet\bullet}H}$ and hence extends to H^{cop}. Let T be the antipode for H^{cop} and S the antipode for H.

Then for $h \in H$, we have $h_{(2)}T(h_{(1)}) = \epsilon(h)1_H = T(h_{(2)})h_{(1)}$ (notice the difference from the normal axiom of an antipode). Applying S and using the fact S is an algebra antihomomorphism, it follows that $(S \circ T)(h_{(1)})S(h_{(2)}) = \epsilon(h)1_H = S(h_{(1)})(S \circ T)(h_{(2)})$. We see $(S \circ T)$ is both a left and a right convolution inverse to S. Thus, $S \circ T = \mathrm{Id}_H$ since the convolution inverse is unique. Replacing h by $S(h)$ in $h_{(2)}T(h_{(1)}) = \epsilon(h)1_H = T(h_{(2)})h_{(1)}$, it follows that $S(h_{(1)})(T \circ S)(h_{(2)}) = \epsilon(h)1_H = (T \circ S)(h_{(1)})S(h_{(2)})$ Thus, $T \circ S = \mathrm{Id}_H$, and we conclude that S is a bijection. ∎

A subclass of examples related to the topology of loop spaces [51] is of the following special diagonal form.

Definition 3.26. A coaugmented coalgebra C over R which has a decomposition into Quillen connected subcoalgebras, $C = \bigoplus_{g \in \mathrm{gpl}(C)} C_g$ with $C_g = Rg \oplus \overline{C_g}$, where Rg is free is called *looplike*. It is a *component coalgebra* in case it is graded and the degree 0 part is $C[X]$.

The name looplike stems from the fact a path coalgebra for a category that is totally disconnected has such a form. Disconnected means that for any two objects $X \neq Y$, $\mathrm{Hom}(X, Y) = \varnothing$ and hence all paths of composable morphisms are loops, viz. they have the same source and target. In the particular case of a quiver, the vertex set V is the set of grouplike elements. In the arrow set of Q there is no arrow between any distinct vertices, i.e., $Q(x, y) = \varnothing$ if $x \neq y \in V$. Note that counterintuitively, even for a looplike coalgebra, $F_{g,h}^{QT}$ is non-zero as $g - h$ is a (g, h)-skew primitive, as mentioned above, see Remark 3.12.

Remark 3.27. This notion of component coalgebra is motivated by $H_*(X, R)$ and $H_*(\Omega X, R)$, where X is a based space. If both modules are R-flat, the first homology ring is a component coalgebra and the second ring is a component Hopf algebra. Moreover, the condition for the second ring to be connected in the coalgebra sense is equivalent to X being connected in the topology sense; cf. [51].

Proposition 3.28. *A looplike coalgebra is a pathlike coalgebra.*

Proof. By definition $\overline{\Delta}_g(c) = \Delta(c) - g \otimes c - c \otimes g$ which when restricted to C_g coincides with the comultiplication on the coaugmented algebra $C_g = Rg \oplus \overline{C_g}$. The latter has $\epsilon_g(g) = 1$, $\eta_g(1) = g$ and $\eta_g \epsilon_g = \phi_g$ as the projection to the first factor. Now as $\Delta(C_g) \subset C_g \boxtimes C_g$ by assumption and hence $\overline{\Delta}_g \colon C_g \to C_g \boxtimes C_{g,g}$ and $C_g \subset F^Q C_g$, since C_g is Quillen connected. The counit is compatible by Lemma 3.9; viz. $\epsilon|_{C_g \subset F^Q C_{g,g}} = \epsilon_g$. Thus, we can identify C_g as lying inside the diagonal part $F^{QT} C_{g,g}$ of the bivariate Quillen filtration. Hence the bivariate Quillen filtration is exhaustive as the C_g already exhaust C. The degree 0 part is $C[X]$ by definition and there are no semigrouplike elements as $\epsilon(g) \neq 0$ for semigrouplikes. $\blacksquare$

4 Renormalization, quotients and localization

The obstruction to having a Hopf algebra structure on a pathlike bialgebra is the invertibility of the grouplike elements. There are basically two approaches to remedy this perceived deficiency. One is adding formal inverses, which is possible by universal constructions. But, as at the end of the day, for renormalization à la Connes–Kreimer, one actually only needs convolution inverses for characters, there is another option. This is to restrict the target algebras or to place restrictions on the characters – for instance that the target algebra is commutative, or that the characters restricted to grouplike elements have special properties. In this case there are universal quotients that the characters factor through. The commutativity assumption is natural. Namely, if the target algebra is not commutative, then the convolution of two characters need not be a character. We first briefly explain the setup to give the motivation for these constructions.

4.1 Recollections on renormalization via characters

A renormalization scheme in the Connes–Kreimer formalization of the BPHZ renormalization [14], is defined on the convolution algebra of a bialgebra with a Rota–Baxter (RB) algebra, see also [17, 20, 28, 50]. Readers not familiar with RB algebras may consult Appendix B.

A Feynman rule is a character in the convolution algebra $\phi \in \mathrm{Hom}_{R-\mathrm{alg}}(B, A)$. The renormalization of ϕ based on a Rota–Baxter (RB)-operator is a pair of characters $\phi_\pm \in \mathrm{Hom}_{R-\mathrm{alg}}(B, A_\pm)$ such that $\phi = \phi_-^{-1} \star \phi_+$, see Appendix B for the definition of the subalgebras $A_\pm$. The character ϕ_+ is then the renormalized Feynman rule. It was shown in [14, 15] that there is a unique such decomposition for the Connes–Kreimer Hopf algebra of graphs and a commutative RB algebra, if ϕ is the exponential of an infinitesimal character. The solution is then given by a recursive formula. This was generalized in [19] for B any connected bialgebra and a character that is the exponential of an infinitesimal character. The following analysis provides the background for a generalization of these results to the case of a bialgebra with a QT-sequence. If B is a Hopf algebra, then the inverse is given by $\phi_-^{-1} = \phi_- \circ S$, see Proposition 2.6, but if A or the character has special properties, then the full assumption of B being Hopf is not necessary.

4.2 Quotients and quantum deformations for grouplike central and invertible characters

We have already seen that taking the somewhat drastic quotient by $F_0^Q(C)$ makes an sg-flavored coalgebra connected. There are, however, intermediary quotients, which make characters with special properties invertible. These are also motivated by the natural examples [13, 14, 15, 20, 50]. We use the notation $1 = 1_B$.

Proposition 4.1. *The ideal $J = ([B, B])$ spanned by the commutators is a coideal. And, if A is commutative, then any character $\phi \in \mathrm{Hom}_{R-\mathrm{alg}}(B, A)$, factors through $B^{\mathrm{com}} = B/([B, B])$.*

Proof. J is a coideal: $\Delta(ab - ba) = \sum a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} - b_{(1)}a_{(1)} \otimes b_{(2)} \otimes a_{(1)} = (a_{(1)}b_{(1)} - b_{(1)}a_{(1)}) \otimes a_{(2)}b_{(2)} + b_{(1)}a_{(1)}(a_{(2)}b_{(2)} - b_{(2)}a_{(2)})$ and $\epsilon(ab - ba) = 0$. Since $\phi \colon I \subset \ker(\phi)$, the statement follows. ∎

Proposition 4.2. *A grouplike normalized character factors through the quotient $B^{\mathrm{red}} = B/I_N$, where I_N is the ideal spanned by $1 - g$ for $g \in \mathrm{gpl}(B)$.*

Proof. We need to check that I_N is a coideal. Indeed, $\Delta(1-g) = 1 \otimes 1 - g \otimes g = (1-g) \otimes (1+g)$ and $\epsilon(1 - g) = 0$, since g is grouplike. Furthermore if ϕ is normalized the $\phi(1 - g) = 0$ so that $I \subset \ker(\phi)$. ∎

NB: $R(g - h) \subset I$ since $g - h = (1 - h) - (1 - g) = g - h$. This is the line which keeps the sum in a pathlike bialgebra from being direct, see Remark 3.12.

There is a further quotient that is of interest which was studied in [24]. Consider the ideal I_C generated by $(ag - ga)$ for $a \in B$, $g \in \mathrm{gpl}(B)$.

Proposition 4.3. *I_C is a coideal and the bialgebra B/I_C and any grouplike central character factors through B/I_C.*

Proof. The fact that I_C is a coideal follows as in Proposition 4.1. For any grouplike central character, $\phi(ag - ga) = 0$, so $\ker(\phi) \supset I_C$. ∎

For a grouplike invertible central character, there is a universal construction which can be viewed as a quantum deformation and gives rise to a Brown type coaction. Consider the algebra $B_{\mathbf{q}} = B[q_g, q_g^{-1}]/K \colon g \in \mathrm{gpl}(B), g \neq 1$, that is the ring of Laurent polynomials with coefficients in the possibly noncommutative B modulo the ideal K generated by $q_g q_h = q_{gh}$, which is well defined as the grouplikes form a monoid. The polynomial rings $B_q = B[q_g]/K$, $q \in \mathrm{gpl}(B)$, $g \neq 1$ is a subbialgebra. The bialgebra B_q can be viewed as a multi-parameter quantum deformation.

Note the q_g lie in the center of $B_{\mathbf{q}}$. Endow $B_{\mathbf{q}}$ with the bialgebra structure, where the q_g, q_g^{-1} are grouplike, then K is a coideal. Consider the ideal I of the Laurent series generated by $q_g - g$ which descends to $B_{\mathbf{q}}$. In the quotient $B_{\mathbf{q}}/I$, the image of the grouplike elements lies in the center.

Proposition 4.4. *I is a coideal for $B_{\mathbf{q}}$ and similarly for its subbialgebra B_q.*

Any grouplike central character can be lifted to B_q and factors through the quotient $B_q/I \simeq B/I_C$.

Any grouplike invertible central character ϕ can be lifted to a character $\hat{\phi}\colon B_{\mathbf{q}} \to A$. Furthermore, $\hat{\phi}$ factors through the quotient $B_{\mathbf{q}}/I$ as $\hat{\phi} = \bar{\phi} \circ \pi$ and embedding $B \to B_{\mathbf{q}} \to B_{\mathbf{q}}/I\colon \bar{\phi}(\pi(b)) = \phi(b)$.

As $q_g \to 1\colon B_{\mathbf{q}} \to B$ and $B_q/I \to B/I_N$.

Proof. By definition $\epsilon(q_g - g) = 1 - 1 = 0$ and $\Delta(q_g - g) = q_g \otimes q_g - g \otimes g = (q_g - g) \otimes (q_g + g)$, so I is a coideal in all cases. The lift is given by $\hat{\phi}(q_g) = \phi(g)$ and consequentially for $B_{\mathbf{q}}$ $\hat{\phi}(q_g^{-1}) = \phi(g)^{-1}$. The other statements follow from $\phi(g) = \hat{\phi}(q_g)$. Note that the image of g in $B_{\mathbf{q}}/I$ lies in the center, since q_g does, which shows the isomorphism $B_q/I \simeq B/I_C$ as exactly these grouplike elements have been made central.

The statement about the limit of $B_{\mathbf{q}}$ is clear for B_q/I; notice that as $q_g \to 1$ all the elements $g \sim 1$ which is the quotient by I_N.

The last statement is straightforward. ∎

Proposition 4.5. *Let B be split pathlike then $\mathscr{B}_{\mathbf{q}}/I \simeq B^{\mathrm{red}} \otimes_R R\big[q_g, q_g^{-1}\big]/K$ and similarly for the polynomial subrings.*

Proof. The correspondence is given by collecting the factors of q_g, respectively q_g^{-1} on the right, as they are central. ∎

This quantum deformation also lets us make the coaction of Brown of Proposition 3.18 more commutative and gives a nice interpretation in terms of polynomials and Laurent series.

Definition 4.6. The central quotient Hopf coaction for a split sg-flavored bialgebra is given by

$$\Delta_B\colon \quad B_{\mathbf{q}}/I \to B_{\mathbf{q}}/I \otimes B^{\mathrm{red}}, \qquad \Delta_B(x) = \sum x_{(1)} \otimes \pi(x_{(2)}),$$

where now both factors are Hopf algebras and π simply sets the factors $q_g = 1$ on the left to 1.

Remark 4.7. For the polynomial version in the operad case, the isomorphism in the proposition was realized as the quotient by the ideal $(|-q)$ in [24, Section 2.6], see also Section 5.4.

4.3   Adding formal inverses

In general, one can formally invert the grouplike elements, this is a universal construction called the group completion if one is speaking of a monoid or the localization if one is inverting a multiplicative subset. In the current R-linear context this becomes a subalgebra.

Given a subalgebra S of an R algebra A, the localization at S, $S^{-1}A$ is an object, together with a morphism $j\colon A \to S^{-1}A$ with the universal property that if $f\colon A \to A'$ is an algebra homomorphism such that for all $s \in S$, $f(s)$ is invertible, then f factors as $g \circ j$, with $g\colon S^{-1}A \to A'$. There is an abstract construction in the general case, but it is not effective if there are no other conditions. It can be realized as a quotient of the free algebra of R-linear alternating words in A and S. Assuming that A unital, set

$$T_R(A \otimes S^{\mathrm{op}}) = R \oplus A \otimes S^{\mathrm{op}} \oplus A \otimes S^{\mathrm{op}} \otimes A \otimes S^{\mathrm{op}} \oplus \cdots,$$

which is an algebra by concatenation. The unit is 1_R, where we identify $R \otimes_R M = M$ for any R-module M. Then $AS^{-1} = T_R(A \otimes S^{\mathrm{op}})/I$, where I is the ideal spanned by be following relations:

1) $1 \otimes \bar{1} = 1_R$, $\bar{1} \otimes 1 = 1_R$ which makes $1 \otimes \bar{1}$ a unit,

2) $s \otimes \bar{s} = 1 \otimes \bar{1}$, which inverts the elements of s,

3) $a \otimes \bar{1} \otimes a' \otimes \bar{s} = aa' \otimes \bar{s}$,

4) $a \otimes \bar{s} \otimes 1 \otimes \bar{s}' = a \otimes \bar{s}\bar{s}'$,

where we write $\bar{s}$ to indicate the factors of S^{op}. In this notation $\overline{s_1 s_2} = \bar{s}_2 \bar{s}_1$ by the definition of the opposite multiplication. We let $\epsilon(a \otimes \bar{s}) := \epsilon(a)\epsilon(s)^{-1}$.

If A is also a bialgebra, S is also semigrouplike, and $\epsilon(S) \subset R^{\times}$ then we define the comultiplication:

$$\Delta(a \otimes \bar{s}) = \sum \left(a^{(1)} \otimes \bar{s}\right) \otimes \left(a^{(2)} \otimes \bar{s}\right).$$

This fixes the comultiplication on the free algebra via the bialgebra equation. Defining the counit as $\epsilon(a \otimes \bar{s}) = \epsilon(a)\epsilon(s)^{-1}$, defines the counit on the free algebra.

Proposition 4.8. *I is a coideal and hence AS^{-1} is a bialgebra. Similarly one can define a left localization $S^{-1}A$ by using $T_R(S^{\text{op}} \otimes A)$.*

Proof. This is a straightforward calculation:

1. $\Delta(1 \otimes \bar{1} - 1_R) = 1 \otimes \bar{1} \otimes 1 \otimes \bar{1} - 1_R \otimes 1_R = (1 \otimes \bar{1} - 1_R) \otimes (1 \otimes \bar{1} + 1_R) \in I \otimes A.$

2. $\Delta(s \otimes \bar{s} - 1_R) = (s \otimes \bar{s}) \otimes (s \otimes \bar{s}) - 1_R = (s \otimes \bar{s} + 1_R) \otimes (s \otimes \bar{s} - 1_R) \in A \otimes I.$

3. $\Delta(a \otimes \bar{1} \otimes a' \otimes \bar{s} - aa' \otimes \bar{s}) = \sum a_{(1)} \otimes \bar{1} \otimes \sum a'_{(1)} \otimes \bar{s} \otimes a_{(2)} \otimes \bar{1} \otimes a'_{(2)} \otimes \bar{s} - \sum a_{(1)} a'_{(1)} \otimes \bar{s} \otimes a_{(2)} a'_{(2)} \otimes \bar{s} \equiv 0 \mod (I \otimes A + A \otimes I).$

4. $\Delta(a \otimes \bar{s} \otimes 1 \otimes \bar{s}' - a \otimes \bar{s}\bar{s}') = \sum a_{(1)} \otimes \bar{s} \otimes 1 \otimes \bar{s}' \otimes a_{(2)} \otimes \bar{s} \otimes 1 \otimes \bar{s}' - a_{(1)} \otimes \bar{s}\bar{s}' \otimes a_2 \otimes \bar{s}\bar{s}' \equiv 0 \mod (I \otimes A + A \otimes I).$

For the counit, it is a straightforward check that $\epsilon(I) \equiv 0$. ∎

4.4 Calculus of fractions

There is a more concise version of this construction if the Ore condition and cancellability are met. This gives a calculus of fraction, similar to the calculus of roofs [26]. The right Ore condition states that

$$\forall \, a \in A, \, s \in S, \quad \exists \, a' \in A, \, s' \in S: \quad as = s'a'.$$

One also needs cancellability, or S-regularity. That is if $as = 0$ or $sa = 0$ then $a = 0$. This means that $sa = sb$ implies $a = b$ as does $as = bs$.

Proposition 4.9. *If these conditions are met any element of AS^{-1} can be written as as^{-1}. Furthermore, there is an injective algebra homomorphisms $j \colon A \to AS^{-1}$, which is universal in the sense that any morphisms $f \colon A \to B$ such that $f(s)$ is invertible has a unique lift to $g \colon AS^{-1} \to B$ such that $f = gj$. Finally, the left and right fractions become isomorphic.*

Proof. See [2]. ∎

Remark 4.10. For a free non-commutative algebra $R\{x, y\}$ and $S = \langle y \rangle$ the condition is not met. For the Weyl algebra $R\{x, y\}/(xy - yx = 1)$ the condition is met [2].

In similar spirit, if the product is the tensor product in a monoidal category, the Ore condition is usually not met, as $X \otimes Y$ is rarely $Y \otimes X$. In the case of interest, where one needs to invert the identities id_X, one would need a Y and a morphism g such that for a given morphism $f \otimes \text{id}_X = \text{id}_Y \otimes g$. In the commutative case this always holds. In particular, it does hold in the symmetric monoidal setting when using the coinvariants B^{iso}, see Section 5.2.

A further route of exploration is to use higher homotopy commutativity, i.e., an E_1 for the Ore condition.

For a grouplike central character of a pathlike coalgebra, one does not need full commutativity of B for the Ore condition, but only that of grouplikes. In this case, one can invert the ideal in B/I_C, where the Ore condition holds. Equivalently one can work with B_q and the quotient by I.

Lemma 4.11. *In the special case of B_q the multiplicative set S spanned by the q_g is Ore, since these elements are central, and S is cancellable. The localization $B_q S^{-1}$ is the Laurent-series ring $B_{\mathbf{q}}$.*

Proof. The cancellability is clear, since a polynomial vanishes if and only if all of its coefficients do. The isomorphisms is given by $\bar{q}_g \to q_g^{-1}$ which exists by the universal property. ∎

Remark 4.12 (Abelian case, Grothendieck construction)**.** In the fully commutative case, this concretely boils down to the Grothendieck construction and localization. Given a monoid $(M, \cdot)$, the Grothendieck construction gives the group completion: $K(M) = M \times M/\sim$ where $(m, n) \sim (m', n')$ if $m \cdot n' = m' \cdot n$. There is an injection $M \to K(M)$ given by $m \mapsto (m, 1)$, with inverses given by $(0, m)$. More generally, if R is a commutative ring and S a multiplicatively closed subset, the localization S^{-1} is given by $S^{-1}R = R \times S/\sim$ where $(r, s) \sim (r', s')$ if there is a $t \in S$ such that $t(rs' - r's) = 0$. Similar constructions can be found in [58].

Remark 4.13 (braided/crossed case)**.** The Ore condition is clearly met when S is central as exploited above. It is more lax, though. It for instance allows for a crossed product types and essentially formalizes bicrossed products. These appear naturally for isomorphisms [21], [41, Section 6.2]. For the example of the algebra of morphisms of a category, the Ore condition is guaranteed if there is a commutative diagram for every pair (a, s):

$$
\begin{array}{ccc}
X & \xrightarrow{\ s\ } & Y \\
{\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\
X' & \xrightarrow{\ s'\ } & Y'.
\end{array}
$$

4.5   Application to pathlike bialgebras

The following results follow in a straightforward fashion from the previous ones.

Theorem 4.14. *Let B be a pathlike bialgebra:*

1. *Every grouplike invertible character has a $\star$-inverse and vice-versa.*

2. *The quotient bialgebra B/I_N of Proposition 4.2 is connected and hence Hopf. In particular, every grouplike normalized character $\phi \in \mathrm{Hom}(B, A)$ when factored through to $\bar{\phi} \in \mathrm{Hom}(B/I_N, A)$ has an inverse computed by $\bar{\phi}^{-1} = \bar{\phi} \circ S$.*

3. *The bialgebra $B_{\mathbf{q}}/I$ is Hopf and the $\star$ inverse of a grouplike central character $\phi \in \mathrm{Hom}(B, A)$ when lifted and factored through $\bar{\bar{\phi}} \in \mathrm{Hom}(B_{\mathbf{q}}/I, A)$ has an inverse computed by $\bar{\bar{\phi}}^{-1} = \bar{\bar{\phi}} \circ S$.*

4. *Let $S = \mathrm{gpl}(B)$, then the left localization BS^{-1} is a Hopf algebra. Moreover if S is satisfies the Ore condition and is cancelable, there is an injection $B \to BS^{-1}$ and $B_q/I = BS^{-1}/J$, where J is the ideal generated by $bg - gb$ with $b \in B$, $g \in \mathrm{gpl}(B)$.*

Proof. The first part is in Theorem 3.21. The second part is an application of the same theorem. The third part then follows from Lemma 4.11 and Proposition 3.16. The last part is straightforward. ∎

5 Colored bialgebras from Feynman categories

Deconcatenation is a way to turn a category or colored monoid into a coalgebra, if certain conditions are met. One way to obtain a bialgebra from a coalgebra C is to consider the free algebra which is the tensor algebra TC and extend Δ using the bialgebra equation. More generally, if the colored monoid has a product structure, that is actually a monoidal structure for the category, one can ask if the bialgebra equation holds for the product $\mu = \otimes$ and the deconcatenation coproduct Δ. The answer is that it does for Feynman categories, see [41], that are suitably decomposition finite, so that Δ is well defined. The axioms for a Feynman category roughly say that the objects are a free $\otimes$-monoid on basic objects and that the morphisms are a free monoid on the so-called basic morphisms, which have $\otimes$ indecomposables as targets. There are two cases: one with symmetries, i.e., a symmetric monoidal structure, and one without. The latter is simpler as the former necessitates to pass to equivalence classes, which also makes the product commutative. Concretely, we will consider the bialgebras $B(\mathcal{F})$ for non-symmetric and $B^{\mathrm{iso}}(\mathcal{F})$ for symmetric Feynman categories. With some conditions, both cases are color coaugmented and hence pathlike by Proposition 3.22 and hence Theorem 4.14 applies.

The grouplike elements correspond to the objects and are represented by their identities id_X in the absence of isomorphisms and by their classes for B^{iso}. Note these are usually not $\otimes$-invertible. For this they would have to lie in $\mathrm{Pic}(\mathcal{F})$, which is not usually the case. It is, however, a natural assumption, that their images under characters are invertible as this happens in concrete applications.

5.1 Feynman categories and gradings

Notation 5.1. $\mathrm{Iso}(\mathcal{F})$ is the subcategory of all objects of $\mathcal{F}$ with their isomorphisms. For a category $\mathcal{V}^\otimes$ will denote the free (symmetric) monoidal category. This is essentially the category of words (with or without symmetries) in $\mathcal{V}$, cf. [35, Section 2.4] for an overview and [31, 49] for details. It comes equipped with a functor $j\colon \mathcal{V} \to \mathcal{C}^\otimes$ and the universal property that any functor $F\colon \mathcal{V} \to \mathcal{C}$ to a (symmetric) monoidal category lifts to $F^\otimes\colon \mathcal{V}^\otimes \to \mathcal{C}$, viz. $F = F^\otimes j$.

The comma category $(F \downarrow G)$ for two functors $F\colon \mathcal{D} \to \mathcal{C}$ and $G\colon \mathcal{E} \to \mathcal{C}$ has as objects (X, Y, ϕ) with $X \in \mathcal{D}$, $Y \in \mathcal{C}$, and $\phi \in \mathrm{Hom}_\mathcal{C}(F(X) \to G(Y))$. The morphisms are given by commutative diagrams. That is a morphism from $(X, Y, \phi) \to (X', Y, \phi')$ is given by a pair $f\colon X \to X'$, $g\colon Y \to Y'$, such that $G(g)\phi = \phi'F(f)$. If F, G are clear, we will write $(\mathcal{D} \downarrow \mathcal{E})$ for the comma category. The category $\mathrm{Iso}(\mathrm{ar}(F))$, viz. the underlying groupoid of the arrow category, has the morphisms of $\mathcal{F}$ as objects while the morphisms of the category are given by pairs of isomorphisms (σ, σ') which map $f \mapsto \sigma^{-1}f\sigma'$ when the compositions are defined.

A slice category is the comma category $(\mathcal{C} \downarrow X)$, where (X, id_x) is a viewed as a subcategory with one object.

Definition 5.2. A Feynman category is a triple $(\mathcal{V}, \mathcal{F}, \imath)$, where $\mathcal{V}$ is groupoid, $\mathcal{F}$ is a (symmetric) monoidal category and $\imath\colon \mathcal{V} \to \mathcal{F}$ is a functor, such that

1. The free (symmetric) monoidal category $\mathcal{V}^\otimes$ is equivalent to $\mathrm{Iso}(\mathcal{F})$ via $\imath^\otimes$.

2. The groupoid of morphisms $\mathrm{Iso}(\mathrm{ar}(\mathcal{F}))$ is equivalent to the free (symmetric) monoidal category on the morphisms from objects from $\mathcal{V}^\otimes$ to $\mathcal{V}$. In formulas $\mathrm{Iso}(\mathrm{ar}(\mathcal{F})) \simeq \mathrm{Iso}(\mathcal{V}^\otimes \downarrow \mathcal{V})^\otimes$.

3. All slice categories $\mathcal{F}$ are essentially small.

A Feynman category is *strict* if its monoidal structure is strict.

The objects of $\mathcal{V}$, sometimes called vertices, are the basic objects in the following sense. The first condition says that every object of $\mathcal{F}$ decomposes as $X \simeq \imath(*_1) \otimes \cdots \otimes \imath(*_n)$ essentially

uniquely, where this means up to isomorphisms on the indecomposables and in the symmetric case up to permutations.

Dropping the i, the second condition means that any morphism $\phi\colon Y \to X$ of $\mathcal{F}$ decomposes into indecomposables as well $\phi = \phi_1 \otimes \cdots \otimes \phi_n$, where each of the $\phi_i\colon Y_i \to *_i$ is a basic morphism, with $X \simeq *_1 \otimes \cdots \otimes *_n$ and $Y \simeq Y_1 \otimes \cdots \otimes Y_n$. Again this is essentially unique, that is up to isomorphisms on the ϕ_i and permutations in the symmetric case. The third condition is technical and used for the existence of certain colimits.

In general there is a natural grading given by $|X| = $ word length. That is if $X \simeq *_1 \otimes \cdots \otimes *_n$ then $|X| = n$. This is well defined due to axiom (i). This give a grading on the morphisms as $|\phi| = |X| - |Y|$. All the algebras/coalgebra structures are graded by $|\cdot|$.

Definition 5.3. A *degree function* on a Feynman category $\mathfrak{F}$ is a map $\deg\colon \mathrm{Mor}(\mathcal{F}) \to \mathbb{N}_0$, such that: $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$ and $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$, and every morphism is generated under composition and monoidal product by those of degree 0 and 1.

For a *weak degree function*, the first condition is relaxed to $\deg(\phi \circ \psi) \geqslant \deg(\phi) + \deg(\psi)$. In addition, a (weak) degree function is called *proper* if $\deg(\phi) = 0$ if and only if ϕ is an isomorphism.

In practice, Feynman categories come with a presentation. This is a set of generators of the basic morphisms together with relations among them, cf. [41, Section 5]. The generators are a set $P(\mathcal{F}) \subset (\mathcal{F} \downarrow \mathcal{V})$ of non-invertible elements such that any morphism in $(\mathcal{F} \downarrow \mathcal{V})$ up to isomorphism can be written as $\hat{\phi}_n \hat{\phi}_{n-1} \cdots \hat{\phi}_2 \phi_1$, where $\phi_i \in P(\mathcal{F})$ and we use the short hand notation $\hat{\phi} = \mathrm{id}_{*_1} \otimes \cdots \otimes \mathrm{id}_{*_{k-1}} \otimes \phi \otimes \mathrm{id}_{*_{k+1}} \otimes \cdots \otimes \mathrm{id}_{*_n}$ with ϕ in the k-th position and the factors of id_{*_i} are identities of some of the basic objects. Note that generators $\phi_i\colon X_i \to *$ can have sources with $|X_i| \geqslant 0$. Commonly the sources can be restricted to $|X_i| \in \{0, 1, 2\}$ or even $\{1, 2\}$. The relations are given by commutative diagrams in the $\hat{\phi}_k$. We call $\mathfrak{F}$ *primitively generated*, if it has a set of generators which are essentially irreducible under composition.

We call a presentation *effective* if for each $\phi \in (\mathcal{V} \downarrow \mathcal{F})$ there is a maximal number $|\phi|_{\max}$ of generators in any decomposition as above. Defining $|\phi_0 \otimes \phi_1|_{\max} := |\phi_1|_{\max} + |\phi_2|_{\max}$ extends $|-|_{\max}$ to all morphisms. The following is straightforward:

Lemma 5.4. *For a Feynman category with an effective presentation $|-|_{\max}$ is a weak degree function. A presentation in which the relations are homogenous in the number of generators is effective and $|-|_{\max}$ is a degree function.*

Remark 5.5. These considerations also work in an enriched setting, that is if the morphisms themselves are objects of a symmetric monoidal category, e.g., R-modules. We will not consider the details of this situation more deeply here and refer to [36].

5.2   Colored bialgebras from Feynman categories

We recall [25, Theorem 1.20]:

Theorem 5.6. *If a locally finite strict monoidal category $\mathcal{F}$ is part of a Feynman non-symmetric category $\mathfrak{F}$, then the algebra structure of $\otimes$ and coalgebra structures of deconcatenation give a bialgebra structure on $B(\mathcal{F})$.*

This bialgebra is neither commutative nor cocommutative in general. If $\mathcal{F}^{\mathrm{op}}$ satisfies the condition of the Theorem, we define $B(\mathcal{F}) := B(\mathcal{F}^{\mathrm{op}})^{\mathrm{cop}}$.

Theorem 5.7. *With the assumptions as in the theorem above, B is colored by its objects and color coaugmented. If $\mathfrak{F}$ additionally has a proper degree function, then it graded. If furthermore there are no isomorphisms except for the identities, it is color nilpotent and color connected. It is then pathlike and Theorem* 4.14 *applies.*

Proof. That the coalgebra is colored and color coaugmented follows from Lemma 2.24. The grading is clear. The assertions in the case that there are no isomorphisms except the identities follow from Propositions 2.24 and 3.22. ∎

We now treat the case with isomorphisms. This will yield commutative but generally non-cocommutative bialgebras. In order to define the coproduct, some modifications are needed to avoid an over-counting of decompositions due to the isomorphisms given by the permutations. Otherwise, the coproduct will not be coassociative, see [41, Example 2.11].

Definition 5.8 ([25]). A weak decomposition of a morphism ϕ is a pair of morphisms (ϕ_0, ϕ_1) for which there exist isomorphisms σ, σ', σ'' such that $\phi = \sigma \circ \phi_1 \circ \sigma' \circ \phi_0 \circ \sigma''$. We introduce an equivalence relation which says that $(\phi_0, \phi_1) \sim (\psi_0, \psi_1)$ if they are weak decompositions of the same morphism. An equivalence class of weak decompositions will be called a decomposition channel which is denoted by $[(\phi_0, \phi_1)]$. $\mathcal{F}$ is called essentially decomposition finite, if the following sum is finite:

$$\Delta^{\mathrm{iso}}([\phi]) = \sum_{[(\phi_0, \phi_1)]} [\phi_0] \otimes [\phi_1], \tag{5.1}$$

where the sum is over a complete system of representatives for the decomposition channels for a fixed representative ϕ.

Note that parallel to (2.1) this is the monoidal version of the coproduct. The categorical version is given by Δ^{op}.

If the category $\mathcal{F}$ is symmetric monoidal, there are extra symmetries permuting objects and morphisms, and the behavior of these automorphism groups is not bialgebraic. Therefore one has to take coinvariants under the action of the automorphism groups of the objects acting on the morphisms. Because the commutators and associators are isomorphisms, the coinvariants are associative, commutative monoids. To take the coinvariants, consider the R-module $R[\mathrm{Mor}(\mathcal{F})]$ with $f \sim g$ if there are isomorphisms σ, σ', s.t. $f = \sigma' f \sigma^{-1}$. Note the product descends since if $\phi \sim \phi'$ and $\psi \sim \psi'$ then $\phi \otimes \psi \sim \phi' \otimes \psi'$. We denote the quotient by $R[\mathrm{Mor}(\mathcal{F})]^{\mathrm{iso}} = R[\mathrm{Mor}(\mathcal{F})]/\sim$.

The following is contained in [41, Theorem 2.1.5]:

Theorem 5.9. *If a channel decomposition finite monoidal category $\mathcal{F}$ is part of a Feynman category $\mathfrak{F}$, then the algebra structure defined by $\otimes$ together with the coalgebra structure given by (5.1) define a bialgebra structure on $R[\mathrm{Mor}(\mathcal{F})]^{\mathrm{iso}}$. The counit is given by $\epsilon([\mathrm{id}_X]) = 1$ and $\epsilon([\phi]) = 0$ if $[\phi] \neq [\mathrm{id}_{s(\phi)}]$.*

If $\mathcal{F}^{\mathrm{op}}$ satisfies the conditions of the Theorem, we define $B^{\mathrm{iso}}(\mathcal{F}) := B^{\mathrm{iso}}(\mathcal{F}^{\mathrm{op}})^{\mathrm{cop}}$.

Note that without changing the coalgebra, we can assume that $\mathcal{F}$ is strict and skeletal.

Lemma 5.10. *In the situation of the theorem above, the semigrouplike elements are the classes of the identities $[\mathrm{id}_X]$ and they are grouplike.*

Proof. Consider the coproduct of any ϕ for which $[\phi] \neq [\mathrm{id}_{s(\phi)}] = [\mathrm{id}_{t(\phi)}]$ has at least two distinct terms

$$\Delta^{\mathrm{iso}}([\phi]) = [\mathrm{id}_{s(\phi)}] \otimes [\phi] + [\phi] \otimes [\mathrm{id}_{t(\phi)}] + \cdots.$$

These are the only two terms precisely if $[\phi]$ is skew-primitive. No such $[\phi]$ is semigrouplike, but, if $[\phi] = [\mathrm{id}_X]$, then these two decomposition channels coincide and $\Delta([\mathrm{id}_X]) = [\mathrm{id}_X] \otimes [\mathrm{id}_X]$ as there are no left or right invertible elements by Lemma 2.24. Since $\epsilon([\phi]) = 1$, these are grouplike. ∎

Using Δ^{iso} thus turns essentially indecomposable morphisms into skew primitives.

Remark 5.11. For a proper (weak) degree function the degree 0 part is free on the grouplikes, which are the classes $[\mathrm{id}_x]$. The degree 1 part is generated by the isomorphisms classes of essentially indecomposable elements, whose classes are skew-primitive. If $\mathcal{F}$ is primitively generated, then the classes of the generators are in degree 1.

Theorem 5.12. *If $\mathcal{F}$ satisfies the assumptions of Theorem 5.9, the bialgebra $B^{\mathrm{iso}}(\mathcal{F})$ is colored, color coaugmented, color conilpotent and color connected and hence pathlike and Theorem 4.14 applies.*

If $\mathfrak{F}$ additionally has a proper degree function, then it is graded, color nilpotent and color connected. Thus, it is pathlike and Theorem 4.14 applies.

Proof. The colors are the isomorphism classes of objects which are grouplikes by Lemma 5.10. It is freely generated as a module by the isomorphism classes of morphisms, so the grouplike elements split off as a direct summand $C[R] = \bigoplus_{x \in R} [\mathrm{id}_x]$, where R is a set of isomorphism classes of objects which can be identified with the set of objects of the skeleton of $\mathcal{F}$.

The left and right coactions are given by $\lambda([\phi]) = [\mathrm{id}_{s(\phi)}] \otimes [\phi]$ and $\rho([\phi]) = [\phi] \otimes [\mathrm{id}_{t(\phi)}]$. This is well defined as can easily be checked. By definition of a composition channel $s(\phi_1) \simeq t(\phi_0)$, so that the decomposition is colored. The coaugmentation comes from the splitting.

Since isomorphisms have degree 0, a proper degree function descends to the isomorphism classes: $\deg([\phi]) := \deg(\phi)$. For the same reason the coproduct (5.1) is graded. As all the isomorphisms are in the classes of the identities, B^{iso} is color reduced and hence conilpotent by Corollary 2.21. As B^{iso} is a free R-module, Lemma 2.23 applies and it is connected. Proposition 3.22 guarantees that B^{iso} is pathlike in this case. ∎

Remark 5.13. In these cases the localization and the Laurent series will give Hopf algebras. The q deformations will have parameters given by the isomorphism classes of basic objects. The $[\mathrm{id}_x]$ are also the classes that will be formally inverted. This can be thought of as extending $\mathcal{F}$ to a localized category $S^{-1}\mathcal{F}$ such that the x live in the Picard subcategory $\mathrm{Pic}\left(S^{-1}\mathcal{F}\right)$.

Remark 5.14. The characters need not be graded, but what is common is that they map the additive degree to a multiplicative one as in $n \mapsto q^n$, where q can be of degree 0, which can also be achieved by shifting degrees or inverting elements in the target. One typical such q would be $\frac{1}{2\pi i}$, see [46], or factors of $\zeta^{\mathrm{m}}(2)$ in the framework of Brown, see [7].

5.3   Details for the main examples

We will give the details for three types of examples, which cover the main applications.

5.3.1   Set based/simplicial examples

The category $\mathcal{F}\mathrm{in}\mathcal{S}\mathrm{et}$ is a Feynman category. There is only one basic object up to isomorphism, which is a one element set. The basic degree function $|X|$ coincides with the cardinality. This is not a proper degree function, but it is proper when restricted to the Feynman subcategory of finite sets and surjections FS. Similarly, $-|\phi|$ is a proper degree function for the Feynman subcategory of finite sets and injections FI, cf. [25]. The category $\mathcal{F}\mathrm{in}\mathcal{S}\mathrm{et}_>$ of ordered finite sets with order preserving maps is a non-symmetric Feynman category, cf. [35]. As before, $|\phi|$ is a degree function which is proper when restricted to the non-symmetric Feynman subcategory OS of ordered finite sets and surjections, and $-|\phi|$ is a degree function for OI, that is ordered sets and injections. The skeleton of $\mathcal{F}\mathrm{in}\mathcal{S}\mathrm{et}_>$ is the augmented simplicial category Δ_+ and the subcategories of surjections and injections restrict as Δ_+^{surj} and Δ_+^{inj}. For a skeletal $\mathcal{V}$ there is only one basic object $[0] = \{0\}$. Its powers can be identifies with $[0]^0 = \varnothing =: [-1]$ and $[0]^{\otimes n} = [n-1] = \{0, \ldots, n-1\}$ – using the traditional simplicial notation.

Proposition 5.15. *The bialgebras* $B\big(\Delta_+^{\mathrm{surj}}\big)$, $B\big(\Delta_+^{\mathrm{inj}}\big)$, $B^{\mathrm{iso}}(\mathrm{FI})$, $B^{\mathrm{iso}}(\mathrm{FS})$ *as well as their coop-posites are all colored connected pathlike bialgebras. The colors can be identified with* $\mathbb{N}_0$. *The only grouplike elements are the* $\mathrm{id}_{[n]}$ *which are monoidally generated by* $\mathrm{id}_{\mathbb{1}}$ *and* $\mathrm{id}_{[0]}$, *respectively their isomorphism classes.*

In the case of surjection the $\sigma_i^n \colon [n+1] \to [n]$ *are* $([n+1], [n])$ *skew primitive* $\mu = \sigma_1^1 \colon [1] \to [0]$, *respectively its class, is a skew primitive generator. In the case of injection the* $\delta_i^n \colon [n-1] \to [n]$ *are* $([n-1], [n])$-*skew primitive and* $\delta_1^0 \colon \varnothing \to [0]$, *respectively its class, is a skew-primitive generator.*

Any character χ *for which* $\chi_{\mathrm{id}_{[0]}}$ *is invertible is invertible. Normalized characters pass through the quotient* B^{red}. *A character is normalized if and only if* $\chi_{\mathrm{id}_{[0]}} = 1$.

In the localized version $S(\mu) = -\mathrm{id}_{[1]}^{-1}\mu\mathrm{id}_{[0]}^{-1}$, *as* $\mathrm{id}_{[1]}^{-1} = \mathrm{id}_{[0]}^{-2}$ *and in Laurent series* $S(\mu) = -q^{-3}\mu$.

Proof. The first set of statements follows from Theorem 5.7 and Theorem 5.12 together with Proposition 3.23. The claims about generation are straightforward. As $\mathrm{id}_{[n]} = \mathrm{id}_{[0]}^{\otimes n-1}$, for any character $\chi(\mathrm{id}_{[n]}) = \chi(\mathrm{id}_{[0]})^{n-1}$ which proves the statements about the characters. The last statement follows from the definitions. ∎

Remark 5.16. By Proposition 3.23 the coopposite bialgebras are pathlike, and Joyal duality, cf. [30], takes on the following form: $B\big(\Delta_+^{\mathrm{surj}}\big)^{\mathrm{cop}} = B\big(\big(\Delta_+^{\mathrm{surj}}\big)^{\mathrm{op}}\big) = B\big(\Delta_{*,*}^{\mathrm{inj}}\big)$, where $\Delta_{*,*}^{\mathrm{inj}} \subset \Delta_{*,*}$ is the subcategory of the double base point preserving injections. It is this bialgebra that is used by Goncharov and Brown, cf. [25, Section 3.6.5] for details. The condition that $\chi_{\mathrm{id}_{[0]}} = 1$ which under Joyal duality reads $\chi(\mathrm{id}_{[1]}) = 1$ is implemented as $I(0, \varnothing, 1) = 1$ in [27], cf. [24, equation (1.5)]. Any such character is group normalized.

Goncharov's symbols $\hat{I}(a_0 : a_1, \ldots, a_n : a_{n+1})$ can be understood as formal characters for a decoration of $\Delta_{*,*}^{\mathrm{inj}}$, see, e.g., [24, Section 1.1.5]. The equation that $\hat{I}(a : \varnothing : b) = 0$ then means that these characters factor through the connected quotient given by the coideal $\mathrm{id}_{\mathbb{1}} - \mathrm{id}_{[1]}$ in $\Delta_{*,*}$ Not insisting on this relation, one can actually assign any invertible value to $\hat{I}(a : \varnothing : b)$, which is then captured by the Laurent series quotient and Brown's coaction. In fact, the symbols of Goncharov are characters with values in a commutative ring, so that one can reduce to B^{com}.

Contrarily to this, the coproduct of Baues, which is also a decoration, is not commutative. The normalization condition in this case is given by insisting that the space be simply connected implemented by collapsing the 1-skeleton, see the discussion in [24].

Remark 5.17. Localizing $\mathrm{id}_{[0]}$, we obtain a Hopf algebra with an invertible antipode. This means that one can construct the Drinfel'd double, which is then a sort of bicrossed product of surjections and double base point preserving injections. This should correspond to the Reedy category on structure on the simplicial category. The precise analysis will appear in future work.

5.4   (Co)operadic examples

The second type of Feynman category is an enrichment $\mathrm{FS}_{\mathcal{O}}$ of FS or $\Delta_{+\mathcal{O}}^{\mathrm{surj}}$ of Δ_+^{surj} as defined in [36, 41]. Note that we can enrich in any monidal category $\mathcal{E}$. For the current purposes this category should be $\mathcal{S}et$ or R-mod, where $R = \mathrm{Hom}(\mathbb{1}_{\mathcal{E}}, \mathbb{1}_{\mathcal{E}})$ is commutative.

Switching to the notation n for the set $\{1, \ldots, n\}$ and setting $\mathrm{Hom}_{\mathrm{FS}_{\mathcal{O}}}(n, 1) = \mathcal{O}(n)$ the second axiom of a Feynman category yields that $\mathrm{Hom}_{\mathrm{FS}_{\mathcal{O}}}(n, m) = \mathcal{O}^{\mathrm{nc}}(n, m)$, where up to isomorphism:

$$\mathcal{O}^{\mathrm{nc}}(n, k) = \bigoplus_{\substack{(n_1, \ldots, n_k) \\ \sum_{i=1}^k n_i = n}} \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k).$$

Here and in the following $\bigoplus$ stands for colimit and we will use $\otimes$ for the monoidal product in $\mathcal{E}$. If the $\mathcal{O}(n)$ are sets, then these are simply $\amalg$ and $\times$. The basic compositions $\mathcal{O}^{\mathrm{nc}}(n,k) \otimes \mathcal{O}^{\mathrm{nc}}(k,1)$ give maps

$$\gamma_{k,n_1,\ldots,,n_k}: \ \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}\Big(\sum n_i = n \Big).$$

In the non-symmetric case, this is the data for a non-symmetric operad and in the symmetric case, the automorphisms $\mathbb{S}_n$ act on the objects yielding a compatible action by pre- and post-composition on the $\mathcal{O}^{\mathrm{nc}}(n,k)$ which is compatible with maps γ and the data is that of a symmetric operad. Vice-versa, specifying a non-symmetric or a symmetric operad fixes an enrichment under the condition that $\mathcal{O}(1)$ splits as $\mathbb{1} \oplus \bar{\mathcal{O}}(1)$, where $\mathbb{1}$ is the component of id_1 and $\bar{\mathcal{O}}(1)$ does not contain any invertible elements, [36, Proposition 3.20]. An operad is called reduced, if $\bar{\mathcal{O}}(1)$ vanishes. The natural grading is $|\mathcal{O}^{\mathrm{nc}}(n,k)| = n - k$.

We will assume that $\mathcal{O}(0) = \varnothing$; if it is not, the considerations of [25, Section 2.11] apply.

Proposition 5.18. *For a non-sigma operad $\mathcal{O}$, with split $\mathcal{O}(1)$ as above, $\Delta^{\mathrm{surj}}_{+\,\mathcal{O}}$ is decomposition finite if and only if $\mathcal{O}(1)$ is decomposition finite. In this case, $B = B\big(\Delta^{\mathrm{surj}}_{+\,\mathcal{O}}\big)$ is a bialgebra. The basic grading induces a grading for B.*

The elements $\mathrm{id}_n = \mathrm{id}_1^{\otimes n}$ are the grouplike elements and there are no other semigrouplike elements. $B(\Delta^{\mathrm{surj}}_{+\,\mathcal{O}})$ is a colored bialgebra, and it is color nilpotent precisely if $C[\mathcal{O}(1)]$ is. In this case, it is color connected and pathlike if it is $\overline{\Delta}$-flat, e.g., if $\mathcal{O}$ is set valued.

A normalized character factors through B^{red} in which all objects are identified. The deformation B_q has one deformation parameter $q = q_{\mathrm{id}_1}$ and a character χ is grouplike invertible if and only if $\chi(\mathrm{id}_1)$ is.

Proof. The first assertion follows from the fact that the natural degree allows to reduce the question of decomposition finiteness and nilpotence to $\mathcal{O}(1)$, cf. [24] for more details.

The next statement follows from Theorem 5.7. The statement of being color nilpotent follows directly from the fact that the degree function is a grading for the coalgebra structure. The final statements are then straightforward. ∎

Analogously one can prove:

Theorem 5.19. *For a symmetric operad $\mathcal{O}$ with split $\mathcal{O}(1)$ as above $\mathrm{FS}_{\mathcal{O}}$ is channel decomposition finite, if and only if $\mathcal{O}(1)$ is. In this case, $B^{\mathrm{iso}}(\mathrm{FS}_{\mathcal{O}})$ is a bialgebra. The basic grading gives a degree function. The elements $[\mathrm{id}_n] = [\mathrm{id}_1]^{\otimes n}$ are the grouplike elements and there are no other semigrouplike elements and $B(\mathrm{FS}_{\mathcal{O}}))$ is a colored bialgebra. It is color nilpotent precisely if its restriction to the isomorphism classes of $\mathcal{O}(1)$ is. In this case, it is color connected and pathlike if is $\overline{\Delta}$-flat, which is the case if $\mathcal{O}$ is set valued.*

A normalized character factors through B^{red} in which all objects are identified. The deformation B_q has one deformation parameter $q = q_{[\mathrm{id}_1]}$ and a character χ is grouplike invertible if and only if $\chi([\mathrm{id}_1])$ is.

Example 5.20 (CK-tree bialgebras)**.** Concretely consider the operad of rooted trees or the non-sigma operad of planar planted trees, where $\mathcal{O}(n)$ are the trees with n leaves and the composition is given by gluing on leaves. Then the coproduct is the familiar Connes–Kreimer type coproduct [13, 22, 34]:

$$\Delta([\tau]) = \sum_{\tau_0 \subset \tau} [\tau_0] \otimes [\tau \backslash \tau_0], \qquad \text{respectively} \quad \Delta(\tau) = \sum_{\tau_0 \subset \tau} \tau_0 \otimes \tau \backslash \tau_0, \tag{5.2}$$

where τ_0 is a rooted subtree – stump – including all its tails and $\tau \backslash \tau_0$ removes all the vertices and half edges of τ_0, viz, the set of cut-off branches, see [24] for details. A summand is depicted
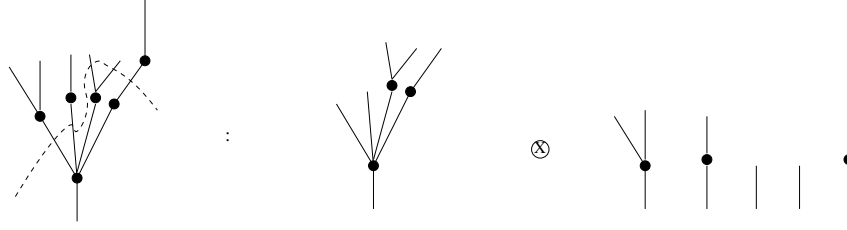
Figure 1. A summand of the coproduct for (planar) trees with leaves and roots. In the planar case, the r.h.s. is either in the order depicted and in the non-planar it is a symmetric product.

in Figure 1. The classes are the $\mathbb{S}_n$ coinvariants which can be thought of as unlabeled. In the symmetric case the r.h.s. is symmetric, that is a commutative product.

This Feynman category is primitively generated by the τ_n which are the trees with n leaves and one vertex. The relations are homogeneous amounting to associativity. The corresponding degree function $|-|_{\max}$ is proper and is equal to the number of vertices. Thus the bialgebras are graded and color connected.

In this language a particular property of the CK-Hopf algebra becomes apparent. It is a pathlike coalgebra whose grouplike elements are generated by $e = \mathrm{id}_{\mathbb{1}}$ and $| = \mathrm{id}_1$ and it has a unique id_1 primitive $\bullet$ which generates a split line $R\bullet$. This yields the special properties explored in [52] and its generalizations, [24, Example 2.50], where, via coloring, many of these are introduced as additional direct components, so that the bialgebras are still pathlike.

In the localized Hopf algebra, $S(\bullet) = -\mathrm{id}_1^{-1} \otimes \bullet \otimes \mathrm{id}_1^{-1}$ and hence in the Laurent series $S(\bullet) = -\bullet q^{-2}$. More generally, for the skew primitive τ_n, $S(\tau_n) = -\mathrm{id}_1^{-1} \otimes \tau_n \otimes \mathrm{id}_n^{-1}$, respectively, $S(\tau_n) = -q^{-(n+1)}\tau_n$. The Brown coaction is given by setting the q terms on the right in Δ to 1.

This explains the fact that the infinitesimal structure can be given by a q-shifted comultiplication and a residue, see [24, Corollary 2.2.5]. This generalizes Brown's infinitesimal structure to the level of operads.

5.5   Feynman categories of graphs

There is a basic Feynman category of graphs called $\mathfrak{G}$, see [41, Section 2.1]. We review salient features here. In the appendix there is a purely set-theoretical version, which is of independent interest and makes things even more combinatorial and hence is conducive to programming.

The category $\mathcal{V}$ has S-labeled corollas $*_S$ as objects with $\mathrm{Aut}(*_S) \simeq \mathrm{Aut}(S)$. This means that a general object is a forest or aggregate of such corollas. The connected version $\mathfrak{G}^{\mathrm{ctd}}$ has morphisms generated by simple edge contractions: $_s\circ_t\colon *_S \amalg *_T \to *_{S\setminus\sigma\amalg T\setminus\{t\}}$ and simple loop contractions $\circ_{ss'}\colon *_S \to *_{S\setminus\{s,s'\}}$. These satisfy quadratic relations and are crossed with respect to the isomorphisms. For any ϕ and any isomorphism σ there are unique ϕ' and σ' such that $\phi \circ \sigma = \sigma' \circ \phi'$. Each morphism has an underlying graph $\Gamma(\phi)$ which is the source corolla, but with one edge for each operation $_s\circ_t$ or $\circ_{st}$. The edges are glued from the half edges s and t. It is defined in a way such that $\Gamma(\phi \circ \sigma) = \Gamma(\phi')$, in the notation above, and its automorphisms are induced from those of the source, see [5, 41]. The isomorphism class of this graph gives the isomorphism class of ϕ: $[\Gamma(\phi)] = [\phi]$.

$\mathfrak{G}^{\mathrm{ctd}} \subset \mathfrak{G}$ is a sub-Feynman category whose basic morphisms have a connected ghost graph. A morphism ϕ in $\mathfrak{G}^{\mathrm{ctd}}$ is basic if and only if $\Gamma(\phi)$ is connected; see Figure 2.

For the non-connected version, there is one more generator $_S\boxminus_T\colon *_S \amalg *_T \to *_{S\amalg T}$. In this case, there is a non-homogeneous relation $_s\circ_t = \circ_{s,t}\,_S\boxminus_T$ which allows one to replace the edge contractions and only retain loop contractions and mergers as generators. The basic morphisms now need not to have connected underlying graph anymore.
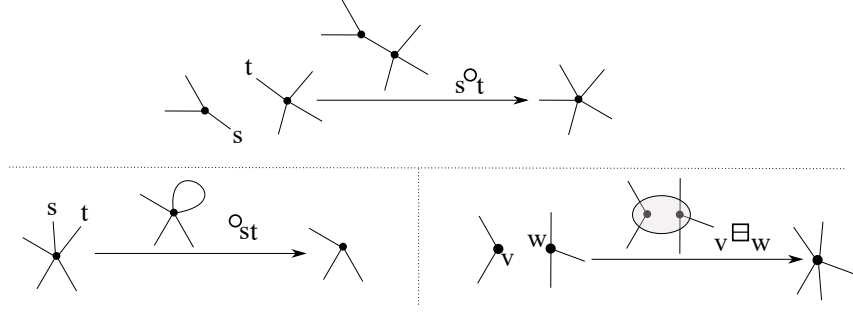
Figure 2. The basic morphisms: Edge contraction loop contraction and merger, whose ghost graph is not connected. The shading is extra data not captured by the ghost graph.

Each corolla $*_S$ is isomorphic to $*_{|S|}$ $-n$ is again short hand for the set $\{1,\ldots,n\}$ – and hence $[\mathrm{id}_{*_S}] = [\mathrm{id}_{|S|}]$. Furthermore fixing isomorphisms $\sigma\colon S \to S$, with $\sigma(s) = 1$, $\sigma(s') = 2$ and $\sigma'\colon T \to T$ with $\sigma'(t) = 1$, $[{}_s\circ_t] = [{}_0\circ_1]$, $[\circ_{s,s'}] = [\circ_{1,2}]$ and $[{}_S\boxminus_T] = {}_{|S|}\boxminus_{|T|}$. Thus there is only one isomorphism class per basic morphism of generating type.

Beside the natural grading $|-|$, which is 1 for an edge contraction or a merger and 0 for an isomorphism or a loop contraction, there is an additional degree given by $\deg(*_S) = |S|$ which is additively continued $\deg(*_S \amalg *_T) = |S| + |T|$, to count the total number of legs in the forest. This yields $\deg(\phi) := \frac{1}{2}(\deg(s(\phi)) - \deg(t(\phi)))$, which is integer, since both loop and edge contractions reduce the set of legs by an even number, while isomorphisms and mergers keep this number fixed. The degree of an isomorphism or a merger is hence 0 and that of a simple loop or edge contraction is 1. In this way $\deg(\phi) = |E(\Gamma(\phi))|$. The sum $wt(\phi) = \deg(\phi) + |\phi|$ is 1 for the mergers and loop contractions and 2 for the edge contractions. This is also the degree $|-|_{\max}$ for the presentation in terms of the ${}_s\circ_t$, $\circ_{ss'}$ and ${}_v\boxminus_w$ and hence is a proper degree function.

Remark 5.21 (loop number and Euler characteristic)**.** In the graphical interpretation for a basic morphism $wt(\phi) = |E(\Gamma(\phi))| + b_0(\phi)$, where b_0 is the number of components of $\Gamma(\phi)$. This is related to the Euler characteristic as follows $\chi(\Gamma(\phi)) = wt(\phi) - |\phi|$, where $\chi(\Gamma(\phi)) = b_0(\Gamma(\phi)) - b_1(\Gamma(\phi)) = |V(\Gamma(\phi))| - |E(\Gamma(\phi))|$ and b_1 is the loop number, aka. first Betti number.

The coproduct in B^{iso} is given by [25, Section 3.6].

$$\Delta[\phi] = \Delta[\Gamma(\phi)] = \sum_{\gamma \subset \Gamma(\phi)} [\gamma] \otimes [\Gamma(\phi)/\gamma]. \tag{5.3}$$

This is a version of the core Hopf algebra [44], but with legs and at a bialgebra level. See Figure 3 for an example of a summand.

Theorem 5.22. *Both $B^{\mathrm{iso}}(\mathfrak{G}^{\mathrm{ctd}})$ and $B^{\mathrm{iso}}(\mathfrak{G})$ are color connected and hence pathlike. They are graded by $\deg, |-|$ and wt, with wt being a proper degree function on $\mathfrak{G}$ and $\deg$ being proper on $\mathfrak{G}^{\mathrm{ctd}}$. For $B^{\mathrm{iso}}(\mathfrak{G}^{\mathrm{ctd}})$ the bivariate Quillen index is given by $\deg$ and for $B^{\mathrm{iso}}(\mathfrak{G})$ the bivariate Quillen index is given by wt.*

More precisely, the grouplike elements are $[\mathrm{id}_{\mathbb{1}}]$ and $[\mathrm{id}_X]$ which are generated by the classes $[\mathrm{id}_{*_S}] = [\mathrm{id}_{|S|}]$. In the connected case, the skew primitive elements are generated by the $[{}_s\circ_t]$ and the $[\circ_{s,s'}]$ while in the connected case these are generated by the $[\circ_{s,s'}]$ and the $[{}_S\boxminus_T]$ via products with grouplikes.
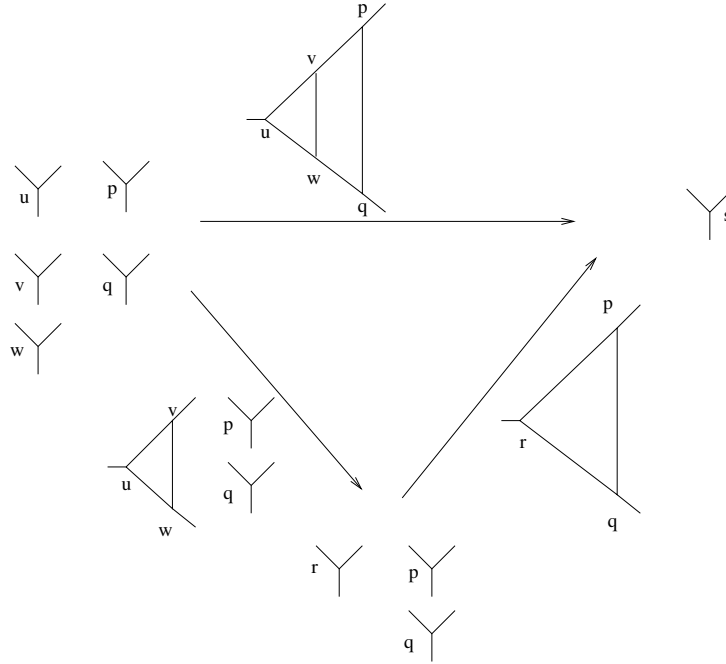
Figure 3. A summand of the coproduct (5.3) in terms of morphisms of aggregates. Note in the actual morphisms all flags and vertices are labeled. In the figure only the vertices are tracked to avoid clutter. For the coproduct only the isomorphism classes of the morphism are involved.

The antipodes for the skew primitives in the localization and the single parameter Laurent polynomials are given as follows:

$$S(\,_n\boxminus_m) = -[\mathrm{id}_n]^{-1}[\mathrm{id}_m]^{-1}[\mathrm{id}_{n+m}]^{-1}\,_n\boxminus_m S(\,_n\boxminus_m) = -q^{-2(n+m)}\,_n\boxminus_m,$$
$$S([\,_s\circ_t]) = [\mathrm{id}_{|S|}]^{-1}[\mathrm{id}_{|T|}]^{-1}[\mathrm{id}_{|S|+|T|-2}]^{-1}[\,_s\circ_t] S[\,_s\circ_t] = -q^{-2(|S|+|T|+1)}\,_s\circ_t,$$
$$S([\circ_{s,s'}]) = [\mathrm{id}_{|S|}]^{-1}[\mathrm{id}_{|T|}^{-1}][\mathrm{id}_{|S|+|T|-2}]^{-1}[\circ_{s,s'}] S[\,_s\circ_t] = -q^{-2(|S|+|T|+1)}[\circ_{s,s'}].$$

The connected Hopf quotient identifies all the objects with $\mathbb{1}$ and hence all grouplikes id_X with $1 = \mathrm{id}_{\mathbb{1}}$. The Brown coaction again sets the q variables on the right to 1, which halves the q degrees in the above formulae.

Proof. The first assertions follows from Theorem 5.12 and the rest is straightforward from the definitions. ∎

Remark 5.23. Besides the core Hopf algebra of all graphs, one can restrict and decorate as is done in [8, 44]. For instance Figure 3 only contains 3-valent vertices, which corresponds to ϕ^3 theory. It is also possible to retract the legs, cf. [24, 25].

6 Conclusion and outlook

6.1 Conclusion

We have shown that pathlike coalgebras in addition to quiver and incidence coalgebras capture the two types of bialgebras stemming from Feynman categories, such as those of Baues, Goncharov and those of Connes–Kreimer. This realization provides several outcomes.

The structure of path-like coalgebras exactly answers the question of what the harbinger of the antipode in the various constructions is. It precisely identifies the grouplike elements as the

obstruction to the existence of the antipode. For characters, their values on theses elements determines their $\star$-invertibility. This also explains the appearance of quotients. These correspond to restrictions on the characters. The most common, which says that all grouplike elements have character value 1, leads to the connected quotient. Other restrictions, like grouplike invertible or grouplike central, yield different quotients and explain the universality of the quantum deformation, which in turn explain the naturally of Brown's coaction.

This sets the stage for further work.

6.2 Outlook

Our analysis is based on using filtrations to reduce to an initial part, where the antipode or convolution inverse is known. The notion of pathlike coalgebras codifies this situation if the initial part is setlike. Section 2.2.4, however, can be taken as a clue that there could be some other structures, which do not necessitate only grouplike elements, but allows for more information about the isomorphisms. The following example of the Drinfel'd double leads the way to a new research in this direction. Physically this corresponds to a (finite) gauge group. In the infinite case, one has to take the precautions spelled out in Appendix A.

In particular, we present a double categorical interpretation in which the antipode basically switches the horizontal and vertical composition. The appearance of double categories is natural in view of [5, 36]. It opens the way to a different version of connectedness, whose lowest degree is of Drinfel'd double type rather than setlike, but keeps a link between the two approaches. We will consider the generalization of the bialgebra and Hopf structures along these lines in [42]. There is a general construction, see [32, Section IX.4] of a Drinfel'd double for a finite dimensional Hopf algebra with invertible antipode; we will concentrate on the case $k[G]$. This is an interesting Hopf algebra whose algebra part comes from a category.

Definition 6.1. For a finite group G the *Drinfel'd double* $D(k[G])$ is the quasi-triangular quasi-Hopf algebra whose underlying *vector space has* the basis $g{\square \atop x}$ with $x, g \in G$ $D(k[G]) = \bigoplus k\, g{\square \atop x}$ with bialgebra structure and antipode given by

$$g{\square \atop x}\; h{\square \atop y} = \delta_{g,xhx^{-1}}\, g{\square \atop xy}, \qquad \Delta(g{\square \atop x}) = \sum_{g_1 g_2 = g} g_1{\square \atop x} \otimes g_2{\square \atop x}, \qquad S(g{\square \atop x}) = x^{-1}g^{-1}x{\square \atop x^{-1}}.$$

The unit is $\sum_{g \in G} g{\square \atop e}$ and the counit is defined by $\epsilon(e{\square \atop x}) = 1$ and $\epsilon(g{\square \atop x}) = 0$ if $g \neq e$.

The algebra structure is the algebra of morphisms of the so-called loop groupoid, $\Lambda\underline{G}$. This is the category which has the object set G and the morphism set $g{\square \atop x} \in \mathrm{Hom}(g, x^{-1}gx)$, see, e.g., [40, 61]. It is hence a colored monoid with colors G with $s(g{\square \atop x}) = g$ and $t(g{\square \atop x} = x^{-1}gx)$.

The coalgebra structure is not the dual, but the categorical coalgebra for the groupoid $\amalg_{x \in G}\underline{G}$ which has $X = G$ as objects and has morphisms $g{\square \atop x} \colon x \to x$. Accordingly, the coalgebra is simply colored for the comodule structures for the set-like coalgebra $k[G] \colon \lambda(g{\square \atop x}) = x \otimes g{\square \atop x}$, $\rho(g{\square \atop x}) = g{\square \atop x} \otimes x$.

The notation $g{\square \atop x}$ goes back to [16], where these are partition functions on a torus for a field theory with finite gauge group. The two indices stand for the monodromies around the fundamental cycles, cf. [33, 40].

Lemma 6.2. *The dual of $D(k[G])$ is a Hopf algebra, whose coproduct is the coalgebra structure coming from the morphisms of a category. In particular, let $\delta_{g{\square \atop x}}$ be the dual basis then*

$$\delta_{g{\square \atop x}}\delta_{h{\square \atop y}} = \delta_{x,y}\, gh{\square \atop x}, \qquad \Delta(\delta_{g{\square \atop x}}) = \sum_y \delta_{g{\square \atop y}} \otimes \delta_{y^{-1}gy{\square \atop y^{-1}x}}, \qquad S(\delta_{g{\square \atop x}}) = \delta_{xg^{-1}x^{-1}{\square \atop x^{-1}}}.$$

Proof. Straightforward computation using the dual basis $\delta_{g\square\atop x}$.   ∎

    The product structure uses the underlying monoid structure of the objects/colors. This is not quite a monoidal structure on the category $\Lambda\underline{G}$. The definition of the antipode uses that the objects (as colors) are invertible. But note, that $g\,{\square\atop x}\otimes h\,{\square\atop y}$ would need to be a morphism from gh to $x^{-1}gxy^{-1}hy$ and if $x=y$ indeed $gh\,{\square\atop x}$ is such a morphism. In general, if G is not Abelian no such morphism needs to exist. The correct notion is that the Drinfel'd double is a subcategory of a natural double category structure on $\mathrm{Iso}(\mathcal{C})$, see [36, Appendix C], where in this particular case $\mathcal{C} = \underline{G}$. This is commensurate with the dictum: "A monoidal category is a two category with one object". The composability of horizontal 2-cells is then exactly the condition that $x=y$ in the above calculation. In particular, the symbols $g\,{\square\atop x}$ are two-morphisms with the following horizontal and vertical composition:

The algebra structure is given by the categorical algebra for the vertical composition, viz. $\circ_h$, as the colored multiplication with $s_v(g\,{\square\atop x}) = t_v(g\,{\square\atop x}) = x$:

$$g\,{\square\atop x}\cdot h\,{\square\atop y} = \delta_{s(g\,{\square\atop x})=t(h\,{\square\atop y})}\, g\,{\square\atop x}\circ_v h\,{\square\atop y}$$

and Δ will be given as the dual of the horizontal composition with $s_h(g\,{\square\atop x}) = g, t_h(g\,{\square\atop x}) = xgx^{-1}$

$$\Delta(g\,{\square\atop x}) = \sum_{(h\,{\square\atop y},k\,{\square\atop z}):\, h\,{\square\atop y}\circ_h k\,{\square\atop z} = g\,{\square\atop x}} h\,{\square\atop y}\otimes k\,{\square\atop z},$$

where we dropped the dual notation δ.

    Another direction of further research is the connection to the cubical structure of the co-product, whose analysis in this context appeared in [24, Section 4.5]. Cubical complexes more generally appear for cubical Feynman categories, [6]. These are intimately related to Cutkosky rules [45] and the work [46]. To illustrate the cubical structure, we recall, cf. [24, equation (4.29)], that simplicial objects have a coproduct whose summands are

$$x_{(0,i_1,\ldots,i_{k-1},n)} \times x_{(0,1,\ldots,i_1)}x_{(i_1,i_1+1,\ldots,i_2)}\cdots x_{(i_{k-1},i_{k-1}+1,\ldots,n)}. \tag{6.1}$$

This has a cubical structure, which can be understood in terms of the Serre diagonal:

$$\delta\colon\ P = [0,1]^{n-1} \xrightarrow{\ \cong\ } \bigcup_{K\cup L=\{1,\ldots,n-1\}} \partial_K^-[0,1]^{n-1}\times\partial_L^+[0,1]^{n-1} \xrightarrow{\ \subset\ } P\times P.$$

The term given in (6.1) is identified with the subset $L = \{i_1,\ldots,i_{k-1}\}$. In view of [46] one has an additional interpretation. The right hand side can be seen as cuts (or stops) in the path from 0 to n on the quiver $[n] = 0\to 1\to\cdots\to n$ and these can be viewed as cuts of a linear graph. This is in line with the simplicial structure of iterated integrals.

Furthermore, there is a symmetry in the Serre diagonal of the cube given by interchanging 0 and 1 in the standard presentation. This switches the coproduct to its opposite. These two observations should generalize to the categorical setting, and when applied to graphs may explain some of the results of [46]. A last aspect is functoriality of the construction, see [25, Section 1.7], which is likely related the structure of cointeracting coalgebras [23, 46] in the present context.

Furthermore, we identified the skew primitive elements (as indecomposables) as being of central importance. This parallels the role of primitives in the usual connected case. In particular, there should be B_+ operators associated to them. These are indeed explicitly present and key elements for E_2-operads and hence for Gerstenhaber brackets and Deligne's conjecture [43]. Moreover, there is a connection between these elements and the terms of the master equations [38] and [41, Section 7.5]. The general structure of B_+ operators will be taken up in [37], where we will also make the connection to Hochschild complexes.

A   Duals, actions and colored algebras

A.1   Duals

The dual of an R module M^* is the R-module $\operatorname{Hom}_R(M, R)$. If M is graded, $M = \bigoplus_{x \in X} M_x$ then the graded dual is $\check{M} = \bigoplus_{x \in X} M_x^*$.

The dual of a coalgebra C is the algebra $C^* := \operatorname{Hom}(C, R)$ with product $fg(c) = (f \otimes g)(\Delta c)$. If C has a counit ϵ_C as a coalgebra then $\epsilon_C \in C^*$ is a unit for the product by definition.

The dual of a *finite* dimensional algebra $A^* = \operatorname{Hom}_R(A, R)$ is a coalgebra with the comultiplication $\Delta(f)(a \otimes b) := f(ab)$. In general, Δ^* is a morphism $A^* \to (A \otimes A)^*$ and there is an injection $A^* \otimes A^* \to (A \otimes A)^*$, which is an isomorphism in the case of a finite dimensional algebra. Therefore, one restricts to the so-called *finite dual* A° which is the maximal subspace of A^* such that $\Delta(A^\circ) \subset A^\circ \otimes A^\circ$. For a Dedekind ring, this is [11]:

$$A^\circ = \{f \in A^* \mid \operatorname{Ker}(f) \text{ contains an ideal } I \text{ for which } A/I \text{ is finitely generated}\}.$$

The Hopf-dual of a Hopf algebra over a field is defined to be H° with the induced structures. For a graded algebra $A = \bigoplus_{x \in X} A_x$, we can consider the graded finite dual $\check{A}^\circ = \bigoplus_{x \in X} X_x^\circ$.

A.2   Colored coalgebras and algebras

If $C[X]$ is the setlike coalgebra on X, $C[X]^*$ has multiplication $fg(x) = f(x)g(x)$ and the constant function $u(x) \equiv 1$ is the unit. In particular, for the dual basis $\delta_x \delta_y = \delta_{x,y} \delta_x$ and $u = \sum_{x \in X} \delta_x$. This is well defined as the application to any $c \in C[X]$ yields only a finite sum. Define the algebra $A[X]$ as the R algebra on the monoid X defined by $xy = \delta_{x,y} y$. This is a subalgebra of the dual algebra $A[X] \subset C^*[X]$ using the identification $x \leftrightarrow \delta_x$. Generators are the functions with finite support on X. $A[X]$ is not unital unless X is finite, but it is almost unital in the sense that $u = \sum_x x \in C[X]^*$ acts as a unit, and when u is restricted to $A[X]$, it acts locally as a finite sum.

If X is finite, this yields a bialgebra structure on $R^{(X)}$. This is a Hopf algebra if and only if X only has one element, as the putative antipode must satisfy $S(x) = x^{-1}$, but the x act as projectors.

Lemma A.1. *If R is a Dedekind ring then $A[X]^\circ = \check{A}[X]^\circ = C[X]$ under the identification $x \leftrightarrow \delta_x$. The coproduct is given by decomposition:*

$$\Delta(\delta_x) = \sum_{(x_1, x_2): \, x_1 x_2 = x} \delta_{x_1} \otimes \delta_{x_2}.$$

Proof. By [11] $f \in A[X]^\circ$ if and only if $A[X]f$ is finitely generated. Since $xf(y) = f(yx) = \delta_{x,y}f(x)$, we see that $f(x)\delta_x \in A[X]f$. As the δ_x are a basis, $A[X]f$ being finitely generated is equivalent to almost all $f(x) = 0$. The coproduct on $\check{A}[X]^\circ$ is given by $\Delta(\delta_x) = \delta_x \otimes \delta_x$ and the identification is again by $x \leftrightarrow \delta_x$. ∎

Linearizing the construction above yields the following definition whose unital version is equivalent to a category enriched in R-Mod.

Definition A.2. A colored algebra is double graded R module $A = \bigoplus_{(x,y)\in X \times X} A_{x,y}$ with R-bilinear associative multiplications $A_{x,z} \otimes A_{z,y} \to A_{x,y}$. It is unital if there are elements $\mathrm{id}_x \in A_{x,x}$ which are left and right units under the multiplication. It is split unital if $A_{x,x} = R\mathrm{id}_x \oplus \bar{A}_{x,x}$.

For a colored monoid the associated colored algebra is given by $A_{x,y} = R^{(M_{x,y})}$ with the induced multiplication. The unital structure is the induced structure.

Suppose M is graded, $M = \bigoplus_{x\in X} M_x$, and hence has a right coaction by $C[X]$. Specializing the general formula for an algebra action induced from a right coalgebra coaction, $f \cdot m = (\mathrm{id} \otimes f)(\rho(m))$, the coaction corresponding to a grading in turn defines an action of $A[X]$ in which the δ_x act by a complete set of orthogonal projectors $\pi_x \colon M \to M_x$, that is $\delta_x m = \pi_x(m)$. The operation $u = \sum_{x\in X} \delta_x$ acts as unit. This sum is finite when acting on any $m \in M$ as M is the direct sum of its graded components.

Lemma A.3. *A colored algebra is an algebra in the category of $A[x]$-bimodules.*

Proof. The multiplication respects the outer grading and taking the product $\otimes_{A[X]}$ amounts to having the inner gradings coincide. ∎

Remark A.4. Note that if one defines a colored algebra as an algebra in $A[x]$-bimodules then one does not obtain finiteness. Indeed $C^*[X]$ is an $A[X]$ bialgebra. To remedy this one can postulate that u acts locally finitely. The question that remains is where the finiteness assumptions should be kept and where they could be relaxed.

B   Rota–Baxter operators and algebras

Definition B.1. A Rota–Baxter (RB) operator of weight λ on an associative R-algebra A a is a linear operator $T \colon A \to A$ that satisfies the equation

$$T(X)T(Y) = T(T(X)(Y)) + T(XT(Y)) + \lambda T(XY). \tag{B.1}$$

A Rota–Baxter (RB) algebra is a unital associative algebra with a choice of Rota–Baxter operator.

Note that if T is an RB-operator, so is $(1 - T)$, furthermore (B.1) says that $\mathrm{Im}(T)$, and similarly $\mathrm{Im}(1 - T)$, are subalgebras of A, which will be called A_+ and A_-. As $(1 - T) + T = 1$, we see that $A = A_+ + A_-$.

NB: This sum does not need to be direct.

The pair (A_+, A_-) is called the *Birkhoff decomposition* of A defined by T.

Remark B.2 (scaling). Scaling the RB operator by μ, we see that μT is a RB algebra of weight $\lambda\mu$, so that if $\lambda \in R^*$ is invertible, we can always scale to have $\lambda = 0$ or $\lambda = -1$.

Example B.3. The standard example are Laurent series $k((x)) = k[[x]][x^{-1}]$, with T being the projection onto the pole part, is an RB-algebra with $\lambda = -1$.

A generalization of this uses projectors, which are operators with $T^2 = 1$. In this case, $1 - T$ is also a projector and the two projectors are orthogonal $T(1 - T) = 0$. For a projector T on an algebra A, set $(1 - T)A = A_+$ and $T(A) = A_-$; then $A = A_+ \oplus A_-$.

Proposition B.4 (projectors and subalgebras). *Given a projector T, if A_+ and A_- are subalgebras, then T is an RB-operator with weight $\lambda = -1$. Furthermore, $T(1)$ is an idempotent, $T(1)^2 = T(1)$, and multiplication by $T(1)$ is a projector onto a subspace of A_-.*

In particular, if $T(1) = 0$, then A_+ is a unital subalgebra, and, if $T(1)$ is regular, then $T(1) = 1$ and A_- is a unital subalgebra.

Proof. The conditions means that $(1-T)$ and T are the projectors onto $A = A_+ \oplus A_-$. Since A_- is a subalgebra, we have that

$$0 = (1 - T)[T(X)T(Y)] = T(X)T(Y) - T(T(X)T(Y)). \tag{B.2}$$

Since A_+ we have that

$$0 = T[(1-T)(X)(1-T)(Y))] = T(XY) - T(T(X)Y) - T(XT(Y)) + T(T(X)T(Y)). \tag{B.3}$$

Plugging (B.2) into (B.3), we obtain (B.1) for $\lambda = -1$. Plugging in $X = Y = 1$ and using that $T^2(1) = T(1)$ we obtain that $T(1)^2 = T(1)$ and hence multiplication by $T(1)$ is a projector. Plugging in $Y = 1$ we obtain $T(1)(1 - T)(X) = 0$. ∎

If A is regular, e.g., a domain, then there are only two possibilities $T(1) = 0$ or $T(1) = 1$. Thus after possibly switching T and $1 - T$, we have that $A = A_+ \oplus A_-$ with A_+ a unital subalgebra and A_- a non-unital subalgebra. Vice-versa an RB-operator for $\lambda = -1$ comes from a subdirect sum.

Definition B.5 ([3]). Two subalgebras (A_-, A_+) form a subdirect sum of A if there exists a subalgebra $C \in A \times B$ which satisfies that for any $a \in A$ there is a unique representation $a = a_- + a_+$, with $(a_-, a_+) \in C$.

Such a decomposition defines $T(a) = a_-$ which satisfies (B.1) for $\lambda = -1$. Vice-versa given an RB T with $\lambda = -1$, setting $A_- = T(A), A_+ = (1 - T)(A), C = \{(T(a), (1 - T)(a)): a \in A\} \subset A_- \times A_+$ gives the subdirect sum.

Theorem B.6 ([3]). *The set of RB-operators T with weight $\lambda = -1$ is in bijection with the subdirect sum decomposition of A under the above correspondence.*

C   Set version of graphs

We give a purely set based, non-graphical version here. A reference for the graphical version is [41, Section 2.1, Appendix A]. The basic objects will be $\mathcal{V} = \mathrm{Iso}(\mathcal{F}\mathrm{in}\mathcal{S}\mathrm{et})$. This fixes the $\mathrm{Iso}(\mathcal{V}^{\otimes})$ to have as objects finite tuples of sets $(S_1, \ldots, S_n)$, where the empty set is allowed both as a tuple and in the entries. The morphisms are componentwise or inner isomorphisms together with outer permutations, which act as follows: A permutation $p \in \mathbb{S}_n$ acts by $p(S_1, \ldots, S_n) = (S_{p(1)}, \ldots, S_{p(n)})$. The inner isomorphisms $\sigma \in \mathrm{Aut}(S_i)$ act as $(\mathrm{id}, \ldots, \mathrm{id}, \sigma, \mathrm{id}, \ldots, \mathrm{id})$. The monoidal product is the joining of tuples. In graphical form, these are aggregates, that is a disjoint unions of corollas. The outer permutations permute the corollas and the inner isomorphisms the flags of the individual corollas.

There are two versions of the category: the connected and the non-connected version. To write the relations in a concise fashion, we abuse notation and write ϕ for $(\mathrm{id}, \ldots, \mathrm{id}, \phi, \mathrm{id}, \ldots, \mathrm{id})$.

For the connected version the morphisms are generated under the monoidal product and concatenation by the isomorphisms and the following morphisms, which called simple edge contractions and simple loop contractions:

$$_s\circ_t\colon\ (S,T)\to(S\backslash\sigma\amalg T\backslash\{t\}),\qquad \circ_{ss'}\colon\ (S)\to(S\backslash\{s,s'\}).$$

These satisfy the following relations.

1. The basic morphisms are equivariant with respect to isomorphisms in the sense that: For $\sigma\in\mathrm{Aut}(S)$, $\sigma'\in\mathrm{Aut}(T)$:

$$_{\sigma(s)}\circ_{\sigma'(t)}(\sigma,\sigma')=\sigma\amalg\sigma'|_{S\backslash\{s\}\amalg T\backslash\{t\}}\ {}_s\circ_t,\qquad \circ_{\sigma(s),\sigma(s')}\sigma=\sigma|_{S\backslash\{s,s'\}}\ \circ_{s,s'}.$$

   For an outer permutation:

$$p\circ_{s,s'}=\circ_{s,s'}p,\qquad p'\,_{p(s)}\circ_{p(t)}=\,_s\circ_t p,$$

   where p' is the image of the permutation in the embedding $\mathbb{S}_{n-1}\to\mathbb{S}_n$ given by the induced block permutation.

2. Loop contractions commute:

$$\circ_{s_1,s_1'}\circ_{s_2,s_2'}=\circ_{s_2,s_2'}\ \circ_{s_1,s_1'}.$$

3. Loop and edge contractions commute:

$$_{s_1}\circ_{t_1}\circ_{s_2 s_2'}=\circ_{s_2 s_2'}\ {}_{s_1}\circ_{t_1}.$$

4. Edge contractions commute if the "edges" do not form a cycle. For $s_1\in S_1$, $t_1\in T_1$, $s_2\in S_2$, $t_2\in T_2$ and $|\{S_1,T_1,S_2,T_2\}|\geqslant 3$

$$_{s_1}\circ_{t_1}\ {}_{s_2}\circ_{t_2}=\,_{s_1}\circ_{t_1}\ {}_{s_2}\circ_{t_2}.$$

5. Edge contractions commute even if they form a cycle, only that after one edge contraction, the second edge contraction is a loop contraction. For $s_1,s_2\in S$, $t_1,t_2\in T$,

$$\circ_{s_2,t_2}\ {}_{s_1}\circ_{t_1}=\circ_{s_1,t_1}\ {}_{s_2}\circ_{t_2}.$$

In the non-connected case, there is an additional generating morphism called simple merger:

$$_S\boxminus_T\colon\ (S,T)\to(S\amalg T).$$

These satisfy the relations:

1. Mergers are equivariant with respect to the internal action

$$_S\boxminus_T(\sigma,\mathrm{id})=(\sigma\amalg\mathrm{id})\,_S\boxminus_T.$$

2. Mergers are equivariant with respect to the outer action. On $(S_1,\dots,S_n)$:

$$_{S_{i+1}}\boxminus_{S_i}=\,_{S_i}\boxminus_{S_{i+1}}\tau_{i,i+1}\qquad\text{and}\qquad p\,_{S_i}\boxminus_{S_{i+1}}=\,_{S_{p'(i)}}\boxminus_{S_{p'(i+1)}}p',$$

   where p' is the image of the permutation p under the embedding $\mathbb{S}_{n-1}\to\mathbb{S}_n$ given by the block permutation with block $(i,i+1)$.

3. Simple mergers commute

$$_{S_1}\boxminus_{T_1}\ {}_{S_2}\boxminus_{T_2}=\,_{S_2}\boxminus_{T_2}\ {}_{S_1}\boxminus_{T_1}.$$

4. Mergers and edge/loop contractions on different sets commute.

5. If $s\in S$ and $t\in T$, then

$$_s\circ_t=\circ_{st}\,_S\boxminus_T.$$

Acknowledgements

References

[1] Andruskiewitsch N., Ferrer Santos W., The beginnings of the theory of Hopf algebras, *Acta Appl. Math.* **108** (2009), 3–17.

[2] Artin M., Noncommutaive rings. Course notes, 1999, available at http://www-math.mit.edu/~etingof/artinnotes.pdf.

[3] Atkinson F.V., Some aspects of Baxter's functional equation, *J. Math. Anal. Appl.* **7** (1963), 1–30.

[4] Baues H.J., The double bar and cobar constructions, *Compositio Math.* **43** (1981), 331–341.

[5] Berger C., Kaufmann R.M., Trees, graphs and aggregates: a categorical perspective on combinatorial surface topology, geometry, and algebra, arXiv:2201.10537.

[6] Berger C., Kaufmann R.M., Derived decorated Feynman categories, in preparation.

[7] Brown F., Motivic periods and the projective line minus three points, arXiv:1407.5165.

[8] Brown F., Feynman amplitudes, coaction principle, and cosmic Galois group, *Commun. Number Theory Phys.* **11** (2017), 453–556, arXiv:1512.06409.

[9] Butcher J.C., An algebraic theory of integration methods, *Math. Comp.* **26** (1972), 79–106.

[10] Cartier P., A primer of Hopf algebras, in Frontiers in Number Theory, Physics, and Geometry. II, Springer, Berlin, 2007, 537–615.

[11] Chen C.Y., Nichols W.D., A duality theorem for Hopf module algebras over Dedekind rings, *Comm. Algebra* **18** (1990), 3209–3221.

[12] Chen K.-T., Iterated integrals of differential forms and loop space homology, *Ann. of Math.* **97** (1973), 217–246.

[13] Connes A., Kreimer D., Hopf algebras, renormalization and noncommutative geometry, *Comm. Math. Phys.* **199** (1998), 203–242, arXiv:hep-th/9808042.

[14] Connes A., Kreimer D., Renormalization in quantum field theory and the Riemann–Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, *Comm. Math. Phys.* **210** (2000), 249–273, arXiv:hep-th/9912092.

[15] Connes A., Kreimer D., Renormalization in quantum field theory and the Riemann–Hilbert problem. II. The β-function, diffeomorphisms and the renormalization group, *Comm. Math. Phys.* **216** (2001), 215–241, arXiv:hep-th/0003188.

[16] Dijkgraaf R., Pasquier V., Roche P., Quasi Hopf algebras, group cohomology and orbifold models, *Nuclear Phys. B Proc. Suppl.* **18** (1990), 60–72.

[17] Duffaut Espinosa L.A., Ebrahimi-Fard K., Gray W.S., A combinatorial Hopf algebra for nonlinear output feedback control systems, *J. Algebra* **453** (2016), 609–643, arXiv:1406.5396.

[18] Dür A., Möbius functions, incidence algebras and power series representations, *Lecture Notes in Math.*, Vol. 1202, Springer-Verlag, Berlin, 1986.

[19] Ebrahimi-Fard K., Guo L., Kreimer D., Integrable renormalization. II. The general case, *Ann. Henri Poincaré* **6** (2005), 369–395, arXiv:hep-th/0403118.

[20] Ebrahimi-Fard K., Kreimer D., The Hopf algebra approach to Feynman diagram calculations, *J. Phys. A: Math. Gen.* **38** (2005), R385–R407, arXiv:hep-th/0510202.

[21] Fiedorowicz Z., Loday J.-L., Crossed simplicial groups and their associated homology, *Trans. Amer. Math. Soc.* **326** (1991), 57–87.

[22] Foissy L., Les algèbres de Hopf des arbres enracinés décorés. II, *Bull. Sci. Math.* **126** (2002), 249–288, arXiv:math.QA/0105212.

[23] Foissy L., Cointeracting bialgebras, Talk at Algebraic Structures in Perturbative Quantum Field Theory, IHES, November 2020.

[24] Gálvez-Carrillo I., Kaufmann R.M., Tonks A., Three Hopf algebras from number theory, physics & topology, and their common background I: operadic & simplicial aspects, *Commun. Number Theory Phys.* **14** (2020), 1–90, arXiv:1607.00196.

[25] Gálvez-Carrillo I., Kaufmann R.M., Tonks A., Three Hopf algebras from number theory, physics & topology, and their common background II: general categorical formulation, *Commun. Number Theory Phys.* **14** (2020), 91–169, arXiv:1607.00196.

[26] Gelfand S.I., Manin Yu.I., Methods of homological algebra, 2nd ed., *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 2003.

[27] Goncharov A.B., Galois symmetries of fundamental groupoids and noncommutative geometry, *Duke Math. J.* **128** (2005), 209–284, arXiv:math.AG/0208144.

[28] Guo L., An introduction to Rota–Baxter algebra, *Surveys of Modern Mathematics*, Vol. 4, International Press, Somerville, MA, Higher Education Press, Beijing, 2012.

[29] Joni S.A., Rota G.-C., Coalgebras and bialgebras in combinatorics, *Stud. Appl. Math.* **61** (1979), 93–139.

[30] Joyal A., On disks, dualities, and Θ categories, Preprint, 1997.

[31] Joyal A., Street R., Braided tensor categories, *Adv. Math.* **102** (1993), 20–78.

[32] Kassel C., Quantum groups, *Graduate Texts in Mathematics*, Vol. 155, Springer-Verlag, New York, 1995.

[33] Kaufmann R.M., Orbifolding Frobenius algebras, *Internat. J. Math.* **14** (2003), 573–617, arXiv:math.AG/0107163.

[34] Kaufmann R.M., On spineless cacti, Deligne's conjecture and Connes–Kreimer's Hopf algebra, *Topology* **46** (2007), 39–88.

[35] Kaufmann R.M., Lectures on Feynman categories, in 2016 MATRIX Annals, *MATRIX Book Ser.*, Vol. 1, Springer, Cham, 2018, 375–438, arXiv:1702.06843.

[36] Kaufmann R.M., Feynman categories and representation theory, in Representations of Algebras, Geometry and Physics, *Contemp. Math.*, Vol. 769, Amer. Math. Soc., Providence, RI, 2021, 11–84, arXiv:1911.10169.

[37] Kaufmann R.M., B-plus operators in field theory, Feynman categories and Hopf algebras, in preparation.

[38] Kaufmann R.M., Manin Yu., Zagier D., Higher Weil–Petersson volumes of moduli spaces of stable n-pointed curves, *Comm. Math. Phys.* **181** (1996), 763–787, arXiv:alg-geom/9604001.

[39] Kaufmann R.M., Medina-Mardones A.M., Cochain level May–Steenrod operations, *Forum Math.* **33** (2021), 1507–1526, arXiv:2010.02571.

[40] Kaufmann R.M., Pham D., The Drinfel'd double and twisting in stringy orbifold theory, *Internat. J. Math.* **20** (2009), 623–657, arXiv:0708.4006.

[41] Kaufmann R.M., Ward B.C., Feynman categories, *Astérisque* **387** (2017), vii+161 pages, arXiv:1312.1269.

[42] Kaufmann R.M., Yang M., Higher categorical Hopf algebras, in preparation.

[43] Kaufmann R.M., Zhang Y., Permutohedral structures on E_2-operads, *Forum Math.* **29** (2017), 1371–1411.

[44] Kreimer D., The core Hopf algebra, in Quanta of Maths, *Clay Math. Proc.*, Vol. 11, Amer. Math. Soc., Providence, RI, 2010, 313–321, arXiv:0902.1223.

[45] Kreimer D., Cutkosky rules from outer space, arXiv:1607.04861.

[46] Kreimer D., Yeats K., Algebraic interplay between renormalization and monodromy, arXiv:2105.05948.

[47] Leroux P., Les catégories de Möbius, *Cah. Topol. Géom. Différ.* **16** (1975), 280–282.

[48] Loday J.-L., Cyclic homology, 2nd ed., *Grundlehren der mathematischen Wissenschaften*, Vol. 301, Springer-Verlag, Berlin, 1998.

[49] Mac Lane S., Categories for the working mathematician, 2nd ed., *Graduate Texts in Mathematics*, Vol. 5, Springer-Verlag, New York, 1978.

[50] Marcolli M., Ni X., Rota–Baxter algebras, singular hypersurfaces, and renormalization on Kausz compactifications, *J. Singul.* **15** (2016), 80–117, arXiv:1408.3754.

[51] May J.P., Ponto K., More concise algebraic topology. Localization, completion, and model categories, *Chicago Lectures in Mathematics*, University of Chicago Press, Chicago, IL, 2012.

[52] Moerdijk I., On the Connes–Kreimer construction of Hopf algebras, in Homotopy Methods in Algebraic Topology (Boulder, CO, 1999), *Contemp. Math.*, Vol. 271, Amer. Math. Soc., Providence, RI, 2001, 311–321, arXiv:math-ph/9907010.

[53] Montgomery S., Hopf algebras and their actions on rings, *CBMS Regional Conference Series in Mathematics*, Vol. 82, Amer. Math. Soc., Providence, RI, 1993.

[54] Munthe-Kaas H.Z., Wright W.M., On the Hopf algebraic structure of Lie group integrators, *Found. Comput. Math.* **8** (2008), 227–257, arXiv:math.AC/0603023.

[55] Nichols W., Sweedler M., Hopf algebras and combinatorics, in Umbral Calculus and Hopf Algebras (Norman, Okla., 1978), *Contemp. Math.*, Vol. 6, Amer. Math. Soc., Providence, R.I., 1982, 49–84.

[56] Quillen D.G., Homotopical algebra, *Lecture Notes in Math.*, Vol. 43, Springer-Verlag, Berlin – New York, 1967.

[57] Rota G.-C., On the foundations of combinatorial theory. I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** (1964), 340–368.

[58] Schmitt W.R., Antipodes and incidence coalgebras, *J. Combin. Theory Ser. A* **46** (1987), 264–290.

[59] Taft E.J., Wilson R.L., On antipodes in pointed Hopf algebras, *J. Algebra* **29** (1974), 27–32.

[60] Takeuchi M., Free Hopf algebras generated by coalgebras, *J. Math. Soc. Japan* **23** (1971), 561–582.

[61] Willerton S., The twisted Drinfeld double of a finite group via gerbes and finite groupoids, *Algebr. Geom. Topol.* **8** (2008), 1419–1457, arXiv:math.QA/0503266.