Kähler–Yang–Mills Equations and Vortices

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Abstract. The Kähler–Yang–Mills equations are coupled equations for a Kähler metric on a compact complex manifold and a connection on a complex vector bundle over it. After briefly reviewing the main aspects of the geometry of the Kähler–Yang–Mills equations, we consider dimensional reductions of the equations related to vortices — solutions to certain Yang–Mills–Higgs equations.

Key words: Kähler–Yang–Mills equations; vortices; gravitating vortices; dimensional reduction; stability

2020 Mathematics Subject Classification: 32Q20; 53C07

Dedicated to Jean-Pierre Bourguignon
with gratitude and admiration

1 Introduction

I would like to single out two contributions of Jean-Pierre Bourguignon related to this paper. The first one is concerned with the study of properties of the scalar curvature of a Riemannian metric [9], more concretely, with the problem of prescribing the scalar curvature in a conformal class (what is referred as the *Nirenberg problem* in the case of the standard conformal class on the 2-sphere); the second contribution regards the study of Yang–Mills connections [10]. In this paper we consider a system of partial differential equations that somehow combines these two problems in a Kählerian set up. These equations, known as the *Kähler–Yang–Mills equations*, were introduced in [1] based on the PhD Thesis of M. García-Fernández (Madrid, 2009) [20]. The Kähler–Yang–Mills equations are coupled equations for a Kähler metric on a compact complex manifold and a connection on a complex vector bundle over it. They emerge from a natural extension of the theories for constant scalar curvature Kähler metrics (Yau–Tian–Donaldson) and Hermite–Yang–Mills connections (Donaldson–Uhlenbeck–Yau).

Fix a holomorphic vector bundle E over a compact complex Kählerian manifold M. The Kähler–Yang–Mills equations intertwine the scalar curvature S_g of a Kähler metric g on M and the curvature F_H of the Chern connection of a Hermitian metric H on E:

$$\mathrm{i}\Lambda_g F_H = \lambda\, \mathrm{Id}, \qquad S_g - \alpha \Lambda^2 \operatorname{tr} F_H^2 = c. \tag{1.1}$$

Here, $\Lambda_g F_H$ is the contraction of the curvature F_H with the Kähler form of g. The equations depend on a coupling constant $\alpha \in \mathbb{R}$, and the constants $\lambda, c \in \mathbb{R}$ are topological.

Equations (1.1) have a symplectic interpretation. As observed by Fujiki [19] and Donaldson [18], the constant scalar curvature condition for a Kähler metric has a moment map interpretation in terms of a symplectic form ω on the smooth compact manifold M. The group of symmetries of the theory for constant scalar curvature Kähler metrics is the group $\mathscr{H}$ of *Hamiltonian symplectomorphisms* of (M, ω). This group acts on the space $\mathscr{J}^i$ of *integrable almost complex structures* on M which are compatible with ω, and this action is Hamiltonian for a natural symplectic form $\omega_{\mathscr{J}}$ on $\mathscr{J}^i$. The moment map interpretation of the Hermite–Yang–Mills equation was pointed out first by Atiyah and Bott [7] for the case of Riemann surfaces and generalized by Donaldson [17] to higher dimensions. Here, one considers the symplectic action of the *gauge group* $\mathscr{G}$ of the Hermitian bundle (E, h) on the space of *unitary connections* $\mathscr{A}$ endowed with a natural symplectic form $\omega_{\mathscr{A}}$. Relying on these two cases, the phase space for the Kähler–Yang–Mills theory is provided by the subspace of the product $\mathscr{P} \subset \mathscr{J}^i \times \mathscr{A}$ defined by the additional integrability condition for a connection $A \in \mathscr{A}$ given by the vanishing of the $(0, 2)$-part of its curvature. Our choice of symplectic structure is the restriction to $\mathscr{P}$ of the symplectic form $\omega_\alpha = \omega_{\mathscr{J}} + \frac{4}{(n-1)!} \alpha \omega_{\mathscr{A}}$, for a non-zero coupling constant $\alpha \in \mathbb{R}$. Here n is the complex dimension of M.

Consider now the *extended gauge group* $\widetilde{\mathscr{G}}$, defined as the group of automorphisms of the Hermitian bundle (E, H) covering Hamiltonian symplectomorphisms of (M, ω). This is a non-trivial extension

$$1 \to \mathscr{G} \to \widetilde{\mathscr{G}} \to \mathscr{H} \to 1, \tag{1.2}$$

where $\mathscr{G}$ is the group of automorphisms of (E, H) covering the identity on M — the usual gauge group —, and $\mathscr{H}$, as above, is the group of Hamiltonian symplectomorphisms of (M, ω). The group $\widetilde{\mathscr{G}}$ acts on $\mathscr{P}$ in a Hamiltonian way for any value of the coupling constant α. In [1], the moment map μ_α is computed, and it is shown that its zero locus corresponds to solutions of (1.1). The coupling between the metric and the connection occurs as a direct consequence of the fact that the extension (1.2) defining the extended gauge group is non-trivial.

It is worth pointing out that extended gauge groups feature in the paper by Bourguignon–Lawson [10], where they are referred to as enlarged gauge groups. In particular they consider the *enlarged gauge group* of a principal bundle P over a compact Riemannian manifold M given by

$$1 \to \mathscr{G}_P \to \widetilde{\mathscr{G}}_P \to \mathscr{I}_M \to 1,$$

where $\mathscr{I}_M$ is the group of isometries of M when the dimension of M is different from 4 and the conformal group when the dimension is 4. As mentioned in [10], a connection on P determines a splitting of the sequence of vector spaces obtained by differentiating the above extension at the identity. This fact is also true in our set-up and plays a crucial role in the computation in [1] of the moment map for the action of $\widetilde{\mathscr{G}}$ on $\mathscr{P}$.

It turns out that equations (1.1) decouple on a compact Riemann surface, due to the vanishing of the first Pontryagin term $\operatorname{tr} F_H^2$, and so in this case the solution to the problem reduces to a combination of the uniformization theorem for Riemann surfaces and the theorem of Narasimhan and Seshadri [16, 34]. For an arbitrary higher-dimensional manifold M, determining whether (1.1) admits solutions is a difficult problem, since in this case these equations are a system of coupled fourth-order fully non-linear partial differential equations.

Despite this, a large class of examples was found in [1] for small α, by perturbing constant scalar curvature Kähler metrics and Hermite–Yang–Mills connections. More concrete and interesting solutions over a polarised threefold — that does not admit any constant scalar curvature Kähler metric — were obtained by Keller and Tønnesen–Friedman [30]. Garcia-Fernandez and Tipler [22] added new examples to this short list, by simultaneous deformation of the complex structures of M and E.

However the problem of finding a general existence theorem for the Kähler–Yang–Mills equations remains pretty much open. In [1], one of the main motivations to study these equations was to find an analytic approach to the algebraic geometric problem of constructing a moduli space classifying pairs (M, E) consisting of a complex projective variety and a holomorphic vector bundle. The stability condition needed for this should naturally be the one solving the Kähler–Yang–Mills equations. In [1], obstructions for the existence of solutions to (1.1) were studied, generalizing the *Futaki invariant*, the *Mabuchi K-energy* and *geodesic stability* that appear in the constant scalar curvature theory. The natural conjecture proposed in [1] is that the existence of solutions of the Kähler–Yang–Mills equations is equivalent to geodesic stability.

To test the above conjecture and provide a new class of interesting examples, inspired by [23], a series of papers [2, 3, 4, 5, 6] have considered the study of dimensional reduction techniques. The simplest situation considered is given by the dimensional reduction of the Kähler–Yang–Mills equations from $M = X \times \mathbb{P}^1$ to a compact Riemann surface X of genus $g(X)$. Here $\mathbb{P}^1$ is the Riemann sphere. In this case, we consider SU(2) acting on M, trivially on X and in the standard way on $\mathbb{P}^1$. We take a holomorphic line bundle L over X, and a holomorphic global section ϕ of L. The pair (L, ϕ) defines an SU(2)-equivariant holomorphic rank 2 vector bundle over $X \times \mathbb{P}^1$ in a canonical way. One can show [2] that an SU(2)-invariant solution to the Kähler–Yang–Mills equations on the bundle E over $X \times \mathbb{P}^1$ is equivalent to having a solution of the equations

$$i\Lambda_g F_h + \frac{1}{2}\big(|\phi|_h^2 - \tau\big) = 0, \qquad S_g + \alpha(\Delta_g + \tau)\big(|\phi|_h^2 - \tau\big) = c \tag{1.3}$$

for a Kähler metric g on X and a Hermitian metric h on L. Here, F_h is the curvature of the Chern connection on L defined by h, $|\phi|_h$ is the pointwise norm of ϕ with respect to h, S_g is the scalar curvature of g, and Δ_g is the Laplacian of the metric on the surface acting on functions. The constant $c \in \mathbb{R}$ is topological, and it can be obtained by integrating (1.3) over X, and τ is a real parameter.

Equations (1.3) are referred as the *gravitating vortex equations* since, in fact, the first equation in (1.3) is the well-known vortex equation of the abelian Higgs model, whose solutions are called *vortices*, and have been extensively studied in the literature in the case of compact Riemann surfaces [11, 23, 25, 36] after the seminal work of A. Jaffe and C. Taubes [28, 37] on the Euclidean plane. In particular, the proof given by S. Bradlow [11] is based on the fact that the vortex equation can be reduced to the Kazdan–Warner equation [29] — an equation that plays a prominent role in the problem studied by Bourguignon–Ezin [9].

It turns out that, when $c = 0$ and $d = c_1(L) > 0$, X is constrained to be the Riemann sphere and the gravitating vortex equations have a physical interpretation, as they are equivalent to the Einstein–Bogomol'nyi equations on a Riemann surface [42, 43]. Solutions of the Einstein–Bogomol'nyi equations are known in the physics literature as *Nielsen–Olesen cosmic strings* [35], and describe a special class of solutions of the abelian Higgs model coupled with gravity in four dimensions [15, 31, 32]. Unlike the cases of genus $g(X) \geq 1$, in genus $g(X) = 0$, new phenomena, not appearing in the classical situation of constant curvature metrics on a surface, arise. Namely, there exist obstructions to the existence of solutions of (1.3), illustrating the fact that, for $g(X) = 0$, the problem of existence of solutions is comparatively closer to the more sophisticated problem of Calabi on the existence of Kähler–Einstein metrics, where algebro-geometric stability obstructions appear on compact Kähler manifolds with $c_1 > 0$.

After presenting the Kähler–Yang–Mills equations in Section 2, and reviewing in the theorems on the existence of solution to the gravitating vortex equations in Section 3, in Section 4, we ponder on the existence of solutions for a non-abelian version of the gravitating vortex equations obtained also by dimensional reduction methods from the Kähler–Yang–Mills equations.

2   The Kähler–Yang–Mills equations

In this section, we briefly explain some basic facts from [1] about the Kähler–Yang–Mills equations, with emphasis on their symplectic interpretation. Throughout this section, manifolds, bundles, metrics, and similar objects are of class C^∞. Let M be a compact symplectic manifold of dimension $2n$, with symplectic form ω and volume form $\mathrm{vol}_\omega = \frac{1}{n!}\omega^n$. Fix a complex vector bundle $\pi\colon E \to M$ of rank r, and a Hermitian metric H on E. Consider the positive definite inner product

$$-\operatorname{tr}\colon\ \mathfrak{u}(r) \times \mathfrak{u}(r) \longrightarrow \mathbb{R}$$

on $\mathfrak{u}(r)$. Being invariant under the adjoint $\mathrm{U}(r)$-action, it induces a metric on the (adjoint) bundle $\operatorname{ad}E_H$ of skew-Hermitian endomorphisms of (E,H). Let Ω^k and $\Omega^k(V)$ denote the spaces of (smooth) k-forms and V-valued k-forms on M, respectively, for any vector bundle V over M. Then, the metric on $\operatorname{ad}E_H$ extends to a pairing on the space $\Omega^\bullet(\operatorname{ad}E_H)$,

$$\Omega^p(\operatorname{ad}E_H) \times \Omega^q(\operatorname{ad}E_H) \longrightarrow \Omega^{p+q},$$

that will be denoted simply $-\operatorname{tr}a_p \wedge a_q$, for $a_j \in \Omega^j(\operatorname{ad}E_H)$, $j = p, q$. An almost complex structure on M compatible with ω determines a metric on M and an operator

$$\Lambda\colon\ \Omega^{p,q} \longrightarrow \Omega^{p-1,q-1} \tag{2.1}$$

acting on the space $\Omega^{p,q}$ of smooth (p,q)-forms, given by the adjoint of the Lefschetz operator $\Omega^{p-1,q-1} \to \Omega^{p,q}\colon \gamma \mapsto \gamma \wedge \omega$. It can be seen that Λ is symplectic, that is, it does not depend on the choice of almost complex structure on M. Its linear extension to adjoint-bundle valued forms will also be denoted $\Lambda\colon \Omega^{p,q}(\operatorname{ad}E_H) \to \Omega^{p-1,q-1}(\operatorname{ad}E_H)$.

Let $\mathscr{J}$ and $\mathscr{A}$ be the spaces of almost complex structures on M compatible with ω and unitary connections on (E,H), respectively; their respective elements will usually be denoted J and A. We will explain now how the Kähler–Yang–Mills equations arise naturally in the construction of the symplectic quotient of a subspace $\mathscr{P} \subset \mathscr{J} \times \mathscr{A}$ of 'integrable pairs'.

The group of symmetries of this theory is the *extended gauge group* $\widetilde{\mathscr{G}}$. Let E_H be the principal $\mathrm{U}(r)$-bundle of unitary frames of (E,H). Then, $\widetilde{\mathscr{G}}$ is the group of automorphisms of E_H which cover elements of the group $\mathscr{H}$ of Hamiltonian symplectomorphisms of (M,ω). There is a canonical short exact sequence of Lie groups

$$1 \to \mathscr{G} \longrightarrow \widetilde{\mathscr{G}} \xrightarrow{\ p\ } \mathscr{H} \to 1, \tag{2.2}$$

where p maps each $g \in \widetilde{\mathscr{G}}$ into the Hamiltonian symplectomorphism $p(g) \in \mathscr{H}$ that it covers, and so its kernel $\mathscr{G}$ is the unitary gauge group of (E,H), that is, the normal subgroup of $\widetilde{\mathscr{G}}$ consisting of unitary automorphisms covering the identity map on M.

There are $\widetilde{\mathscr{G}}$-actions on $\mathscr{J}$ and $\mathscr{A}$, which, combined, give an action on the product $\mathscr{J} \times \mathscr{A}$,

$$g(J, A) = (p(g)J, gA).$$

Here, $p(g)J$ denotes the push-forward of J by $p(g)$. To define the $\widetilde{\mathscr{G}}$-action on $\mathscr{A}$, we view the elements of $\mathscr{A}$ as G-equivariant splittings $A\colon TE_H \to VE_H$ of the short exact sequence

$$0 \to VE_H \longrightarrow TE_H \longrightarrow \pi^*TM \to 0,$$

where $VE_H \subset TE_H$ is the vertical bundle on E_H. Then, the $\widetilde{\mathscr{G}}$-action on $\mathscr{A}$ is $gA := g \circ A \circ g^{-1}$, where $g\colon TE \to TE$ denotes the infinitesimal action on the right-hand side.

For each unitary connection A, we write $A^{\perp}y$ for the corresponding horizontal lift of a vector field y on M to a vector field on E_H. Then, each $A \in \mathscr{A}$ determines a vector-space splitting of the Lie-algebra short exact sequence

$$0 \to \operatorname{Lie}\mathscr{G} \longrightarrow \operatorname{Lie}\widetilde{\mathscr{G}} \xrightarrow{p} \operatorname{Lie}\mathscr{H} \to 0$$

associated to (2.2), because $A^{\perp}\eta \in \operatorname{Lie}\widetilde{\mathscr{G}}$ for all $\eta \in \operatorname{Lie}\mathscr{H}$. Note also that the equation

$$\eta_{\varphi}\lrcorner\,\omega = \mathrm{d}\varphi$$

determines an isomorphism between the space $\operatorname{Lie}\mathscr{H}$ of Hamiltonian vector fields on M and the space $C_0^{\infty}(M,\omega)$ of smooth functions φ such that $\int_M \varphi\,\mathrm{vol}_{\omega} = 0$, where $\mathrm{vol}_{\omega} := \frac{1}{n!}\omega^n$.

The spaces $\mathscr{J}$ and $\mathscr{A}$ have $\widetilde{\mathscr{G}}$-invariant symplectic structures $\omega_{\mathscr{J}}$ and $\omega_{\mathscr{A}}$ induced by ω, that, combined, define a symplectic form on $\mathscr{J} \times \mathscr{A}$, for each non-zero real constant α, given by

$$\omega_{\alpha} = \omega_{\mathscr{J}} + \frac{4\alpha}{(n-1)!}\omega_{\mathscr{A}}. \tag{2.3}$$

The following result provides the starting point for the theory of the Kähler–Yang–Mills equations. This result builds on the moment map interpretation of the constant scalar curvature equation for a Kähler metric, due to Fujiki [19] and Donaldson [18], and the classical result of Atiyah and Bott [7].

Proposition 2.1 ([1]). *The $\widetilde{\mathscr{G}}$-action on $(\mathscr{J} \times \mathscr{A}, \omega_{\alpha})$ is Hamiltonian, with $\widetilde{\mathscr{G}}$-equivariant moment map $\mu_{\alpha}\colon \mathscr{J} \times \mathscr{A} \to (\operatorname{Lie}\widetilde{\mathscr{G}})^*$ given by*

$$\langle \mu_{\alpha}(J,A), \zeta \rangle = 4\mathrm{i}\alpha \int_M \operatorname{tr} A\zeta \wedge (\mathrm{i}\Lambda F_A - \lambda\operatorname{Id})\,\mathrm{vol}_{\omega}$$
$$- \int_M \varphi\left(S_J - \alpha\Lambda^2\operatorname{tr} F_A \wedge F_A - 4\mathrm{i}\lambda\alpha\Lambda\operatorname{tr} F_A\right)\mathrm{vol}_{\omega}$$

for any $\zeta \in \operatorname{Lie}\widetilde{\mathscr{G}}$ covering $\eta_{\varphi} \in \operatorname{Lie}\mathscr{H}$, with $\varphi \in C_0^{\infty}(M,\omega)$.

Here, F_A is the curvature of A, $\lambda \in \mathbb{R}$ is determined by the topology of the bundle and the cohomology class $[\omega] \in H^2(M,\mathbb{R})$, and S_J is the Hermitian scalar curvature of J. Explicitly,

$$F_A = -A\big[A^{\perp}\cdot, A^{\perp}\cdot\big] \in \Omega^2(\operatorname{ad} E_H), \qquad \lambda = \frac{2\pi n c_1(E) \cdot [\omega]^{n-1}}{r[\omega]^n},$$

with the convention $2\pi c_1(E) = [\mathrm{i}\operatorname{tr} F_A]$. A key observation in [1, 20] is that the space $\mathscr{J} \times \mathscr{A}$ has a (formally integrable) complex structure $\mathbf{I}$ preserved by the $\widetilde{\mathscr{G}}$-action, given by

$$\mathbf{I}_{|(J,A)}(\gamma, a) = (J\gamma, -a(J\cdot)) \qquad \text{for } (\gamma, a) \in T_J\mathscr{J} \oplus T_A\mathscr{A}.$$

For positive α, $\mathbf{I}$ is compatible with the family of symplectic structures (2.3), and so it defines Kähler structures on $\mathscr{J} \times \mathscr{A}$. The condition $\alpha > 0$ will be assumed in the sequel.

Suppose now that there exist Kähler structures on M with Kähler form ω. This means the subspace $\mathscr{J}^i \subset \mathscr{J}$ of integrable almost complex structures compatible with ω is not empty. For each $J \in \mathscr{J}^i$, let $\mathscr{A}_J^{1,1} \subset \mathscr{A}$ be the subspace of connections A with $F_A \in \Omega_J^{1,1}(\operatorname{ad} E_H)$, where $\Omega_J^{p,q}$ is the space of (p,q)-forms with respect to J. Then the space of *integrable pairs*

$$\mathscr{P} \subset \mathscr{J} \times \mathscr{A},$$

consisting of elements (J, A) with $J \in \mathscr{J}^i$ and $A \in \mathscr{A}_J^{1,1}$, is a $\widetilde{\mathscr{G}}$-invariant (possibly singular) Kähler submanifold. The zero locus of the induced moment map μ_α for the $\widetilde{\mathscr{G}}$-action on $\mathscr{P}$ corresponds precisely to the solutions of the (coupled) *Kähler–Yang–Mills equations*

$$\mathrm{i}\Lambda F_A = \lambda \, \mathrm{Id}, \qquad S_J - \alpha \Lambda^2 \, \mathrm{tr} \, F_A \wedge F_A = c. \tag{2.4}$$

Here, S_J is the scalar curvature of the metric $g_J = \omega(\cdot, J \cdot)$ and the constant $c \in \mathbb{R}$ depends on α, the cohomology class of ω and the topology of M and E (see [1, Section 2]).

One can express the Kähler–Yang–Mills equations from an alternative point of view in which we fix a compact complex manifold X of dimension n, a Kähler class $\Omega \in H^{1,1}(X)$ and a holomorphic vector bundle E over X. Then these equations, for a fixed constant parameter $\alpha \in \mathbb{R}$, are

$$\mathrm{i}\Lambda_\omega F_H = \lambda \, \mathrm{Id}, \qquad S_\omega - \alpha \Lambda_\omega^2 \, \mathrm{tr} \, F_H \wedge F_H = c, \tag{2.5}$$

where the unknowns are a Kähler metric on X with Kähler form ω in Ω, and a Hermitian metric H on E. In this case, F_H is the curvature of the Chern connection A_H of H on E, and S_ω is the scalar curvature of the Kähler metric. Note that the operator in (2.1) depends on ω, and the constant $c \in \mathbb{R}$ depends on α, Ω and the topology of X and E.

3   The gravitating vortex equations

Let X be a compact connected Riemann surface of arbitrary genus. Let L be a holomorphic line bundle over X and $\phi \in H^0(X, L)$ a holomorphic section of L. We fix a parameter $0 < \tau \in \mathbb{R}$, a coupling constant $\alpha \in \mathbb{R}$, and a real parameter c.

The *gravitating vortex equations*, for a Kähler metric on X with Kähler form ω and a Hermitian metric h on L, are

$$\mathrm{i}\Lambda_\omega F_h + \frac{1}{2}\big(|\phi|_h^2 - \tau\big) = 0, \qquad S_\omega + \alpha(\Delta_\omega + \tau)\big(|\phi|_h^2 - \tau\big) = c. \tag{3.1}$$

Here, S_ω is the scalar curvature of ω, F_h stands for the curvature of the Chern connection of h, $|\phi|_h^2$ is the smooth function on X given by the norm-square of ϕ with respect to h and Δ_ω is the Laplace operator for the metric ω, defined by

$$\Delta_X f = 2\mathrm{i}\Lambda_\omega \bar\partial \partial f \qquad \text{for } f \in C^\infty(\Sigma).$$

We will now show how to derive the gravitating vortex equations (3.1) as a dimensional reduction of the Kähler–Yang–Mills equations (2.5). To do this, we associate to (X, L, ϕ) a rank 2 holomorphic vector bundle E over $X \times \mathbb{P}^1$. This is given as an extension

$$0 \to p^*L \longrightarrow E \longrightarrow q^*\mathcal{O}_{\mathbb{P}^1}(2) \to 0, \tag{3.2}$$

where p and q are the projections from $X \times \mathbb{P}^1$ to X and $\mathbb{P}^1$ respectively. By $\mathcal{O}_{\mathbb{P}^1}(2)$ we denote as usual the holomorphic line bundle with Chern class 2 on $\mathbb{P}^1$, isomorphic to the holomorphic tangent bundle of $\mathbb{P}^1$. Extensions as above are parametrised by

$$H^1(X, p^*L \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(X, L) \otimes H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)\big) \cong H^0(X, L),$$

and we choose E to be the extension determined by ϕ. Let $\mathrm{SU}(2)$ act trivially on X, and in the standard way on $\mathbb{P}^1 \cong \mathrm{SU}(2)/\mathrm{U}(1)$. This action can be lifted to trivial actions on E and p^*L and the standard action on $\mathcal{O}_{\mathbb{P}^1}(2)$. Since the induced actions on $H^0(X, L)$ and $H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)\big) \cong H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\big)^* \cong \mathbb{C}$ are trivial, E is an $\mathrm{SU}(2)$-equivariant holomorphic vector bundle over $X \times \mathbb{P}^1$.

For $\tau \in \mathbb{R}_{>0}$, consider the SU(2)-invariant Kähler metric on $X \times \mathbb{P}^1$ whose Kähler form is $\omega_\tau = p^*\omega + \frac{4}{\tau}q^*\omega_{\mathrm{FS}}$, where ω is a Kähler form on X and ω_{FS} is the Fubini–Study metric on $\mathbb{P}^1$, given in homogeneous coordinates by

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}\mathrm{d}z \wedge \mathrm{d}\overline{z}}{\left(1 + |z|^2\right)^2}$$

and such that $\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = 2\pi$. Assuming that the coupling constants α in (3.1) and (2.5) coincide, we have the following [2].

Proposition 3.1. *The triple (X, L, ϕ) admits a solution (ω, h) of the gravitating vortex equations* (3.1) *with parameter τ if and only if $(X \times \mathbb{P}^1, E)$ admits an SU(2)-invariant solution of the Kähler–Yang–Mills equations* (2.5) *with Kähler form $\omega_\tau = p^*\omega + \frac{4}{\tau}q^*\omega_{\mathrm{FS}}$.*

For a fixed Kähler metric ω, the first equation in (3.1) corresponds to the abelian vortex equation

$$\mathrm{i}\Lambda_\omega F_h + \frac{1}{2}\left(|\phi|_h^2 - \tau\right) = 0, \tag{3.3}$$

for a Hermitian metric h on L. In [11, 23, 25, 36], Noguchi, Bradlow and the author gave independently and with different methods a complete characterisation of the existence of *abelian vortices* on a compact Riemann surface, that is, of solutions of equations (3.3).

Theorem 3.2 ([11, 23, 25]). *Assume that ϕ is not identically zero. For every fixed Kähler form ω, there exists a unique solution h of the vortex equations* (3.3) *if and only if*

$$c_1(L) < \frac{\tau \operatorname{Vol}_\omega(X)}{4\pi}. \tag{3.4}$$

Inspired by work of Witten [41] and Taubes [38], the method in [23] exploited the dimensional reduction of the Hermitian–Yang–Mills equations from four to two dimensions, combined with the theorem of Donaldson, Uhlenbeck and Yau [17, 39, 40].

The constant $c \in \mathbb{R}$ is topological, and is explicitly given by

$$c = \frac{2\pi(\chi(X) - 2\alpha\tau c_1(L))}{\operatorname{Vol}_\omega(X)}, \tag{3.5}$$

as can be deduced by integrating the equations. The gravitating vortex equations for $\phi = 0$, are equivalent to the condition that ω be a constant scalar curvature Kähler metric on X and h be a Hermite–Einstein metric on L. By the uniformisation theorem for Riemann surfaces, the existence of these 'trivial solutions' reduces by Hodge theory to the condition $c_1(L) = \tau \operatorname{Vol}_\omega(X)/4\pi$.

Excluding this trivial case, the sign of c plays an important role in the existence problem for the gravitating vortex equations. The dependence of the gravitating vortex equations (3.1) on the topological constant c is better observed using a Kähler–Einstein type formulation. Using that X is compact, (3.1) reduces to a second-order system of PDE. To see this, we fix a constant scalar curvature metric ω_0 on X and the unique Hermitian metric h_0 on L with constant $\Lambda_{\omega_0}F_{h_0}$, and apply a conformal change to h while changing ω within its Kähler class. Equations (3.1) for $\omega = \omega_0 + dd^c v$, $h = \mathrm{e}^{2f}h_0$, with $v, f \in C^\infty(\Sigma)$, are equivalent to the following semi-linear system of partial differential equations (cf. [2, Lemma 4.3])

$$\Delta f + \frac{1}{2}\left(\mathrm{e}^{2f}|\phi|^2 - \tau\right)\mathrm{e}^{4\alpha\tau f - 2\alpha\mathrm{e}^{2f}|\phi|^2 - 2cv} = -c_1(L),$$
$$\Delta v + \mathrm{e}^{4\alpha\tau f - 2\alpha\mathrm{e}^{2f}|\phi|^2 - 2cv} = 1. \tag{3.6}$$

Here, Δ is the Laplacian of the fixed metric ω_0, normalised to have volume 2π and $|\phi|$ is the pointwise norm with respect to the fixed metric h_0 on L. Note that $\omega = (1 - \Delta v)\omega_0$ implies $1 - \Delta v > 0$, which is compatible with the last equation in (3.6).

For $c \geq 0$, the existence of gravitating vortices forces the topology of the surface to be that of the 2-sphere, because $c_1(L) > 0$ implies $\chi(\Sigma) > 0$ by (3.5). When c in (3.5) is zero, the gravitating vortex equations (3.1) turn out to be a system of partial differential equations that have been extensively studied in the physics literature, known as the *Einstein–Bogomol'nyi equations*. As observed by Yang [44, Section 1.2.1], the existence of solutions in this situation with $\alpha > 0$ constrains the topology of X to be the complex projective line (or 2-sphere) $\mathbb{P}^1$, since $c = 0$ if and only if

$$\chi(X) = 2\alpha\tau c_1(L).$$

We are assuming that $\tau > 0$ and $c_1(L) > 0$.

In the case $c = 0$, for $L = \mathcal{O}_{\mathbb{P}^1}(N)$ and $\mathrm{e}^{2u} = 1 - \Delta v$ the system (3.6) reduces to a single partial differential equation

$$\Delta f + \frac{1}{2}\mathrm{e}^{2u}\big(\mathrm{e}^{2f}|\phi|^2 - \tau\big) = -N, \tag{3.7}$$

for a function $f \in C^\infty\big(\mathbb{P}^1\big)$, where

$$u = 2\alpha\tau f - \alpha\mathrm{e}^{2f}|\phi|^2 + c',$$

and c' is a real constant that can be chosen at will. By studying the *Liouville type equation* (3.7) on $\mathbb{P}^1$, Yang [44, 45] proved the existence of solutions of the Einstein–Bogomol'nyi equations under certain numerical conditions on the zeros of ϕ, to which he refers as a "technical restriction" [44, Section 1.3]. It turns out that these conditions have a precise algebro-geometric meaning in the context of Mumford's geometric invariant theory (GIT) [33], as a consequence of the following result.

Proposition 3.3 ([33, Chapter 4, Proposition 4.1]). *Consider the space of effective divisors on $\mathbb{P}^1$ with its canonical linearised $\mathrm{SL}(2, \mathbb{C})$-action. Let $D = \sum_j n_j p_j$ be an effective divisor, with finitely many different points $p_j \in \mathbb{P}^1$ and integers $n_j > 0$ such that $N = \sum_j n_j$. Then*

(1) *D is stable if and only if $n_j < \frac{1}{2}N$ for all j;*
(2) *D is strictly polystable if and only if $D = \frac{1}{2}Np_1 + \frac{1}{2}Np_2$, where $p_1 \neq p_2$ and N is even;*
(3) *D is unstable if and only if there exists $p_j \in D$ such that $n_j > \frac{1}{2}N$.*

Using Proposition 3.3, Yang's existence theorem has the following reformulation, where "GIT polystable" means either conditions (1) or (2) of Proposition 3.3 are satisfied, and

$$D = \sum_j n_j p_j$$

is the effective divisor on $\mathbb{P}^1$ corresponding to a pair (L, ϕ), with $N = \sum_j n_j = c_1(L)$.

Theorem 3.4 (Yang's existence theorem). *Assume that $\alpha > 0$ and that (3.4) holds. Then, there exists a solution of the Einstein–Bogomol'nyi equations on $\big(\mathbb{P}^1, L, \phi\big)$ if D is GIT polystable for the linearised $\mathrm{SL}(2, \mathbb{C})$-action on the space of effective divisors.*

The converse to Theorem 3.4 is given in [4, 5].

Theorem 3.5. *If $\big(\mathbb{P}^1, L, \phi\big)$ admits a solution of the gravitating vortex equations with $\alpha > 0$, then (3.4) holds and the divisor D is polystable for the $\mathrm{SL}(2, \mathbb{C})$-action.*

Remark 3.6. Notice that this theorem is more general than being a converse to Theorem 3.4 since it does not assume that $c = 0$, and deals with the general gravitating vortex equations (3.1) and not just with the Einstein–Bogomol'nyi equation.

Combining now Theorems 3.4 and 3.5, we obtain a correspondence theorem for the Einstein–Bogomol'nyi equations.

Theorem 3.7. *A triple $\left(\mathbb{P}^1, L, \phi\right)$ with $\phi \neq 0$ admits a solution of the Einstein–Bogomol'nyi equations with $\alpha > 0$ if and only (3.4) holds and the divisor D is polystable for the $\mathrm{SL}(2, \mathbb{C})$-action.*

Another result, conjectured by Yang and proved in [4] is the following.

Theorem 3.8. *There is no solution of the Einstein–Bogomol'nyi equations for N strings superimposed at a single point, that is, when $D = Np$.*

An existence theorem for the gravitating vortex equations (3.1) with $c > 0$ for a triple $\left(\mathbb{P}^1, L, \phi\right)$ with $\phi \neq 0$, similar to Theorem 3.7, is obtained combining Theorem 3.5 with the converse direction in this situation, proved by Garcia-Fernandez–Pingali–Yao [21].

In genus $g(X) = 1$, the gravitating vortex equations (3.1) (with $\phi \neq 0$) always have a solution in the weak coupling limit $0 < \alpha \ll 1$ (see [2, Theorem 4.1] for a precise formulation), and it is an interesting open problem to find effective bounds for α for which (3.1) admit solutions.

Paper [4] deals also with the existence theorem of solutions of (3.1) for surfaces of genus $g \geq 2$, for which one has the following.

Theorem 3.9. *Let X be a compact Riemann surface of genus $g \geq 2$, and L a holomorphic line bundle over X of degree $N > 0$ equipped with a holomorphic section $\phi \neq 0$. Let τ be a real constant such that $0 < N < \tau/2$. Define*

$$\alpha_* := \frac{2g - 2}{2\tau(\tau/2 - N)} > 0. \tag{3.8}$$

Then, the set of α for which (1.3) admits smooth solutions of volume 2π is open and contains the closed interval $[0, \alpha_*]$. Furthermore, the solution is unique for $\alpha \in [0, \alpha_*]$.

This shall be compared with the classical uniformization theorem which establishes that a compact Riemann surface admits a metric of constant curvature with fixed volume, unique up to biholomorphisms. The proof of Theorem 3.9 involves the continuity method, where openness is proven using the moment-map interpretation, while closedness needs *a priori* estimates as usual. The hardest part is the C^0 estimate, and in fact it is for this estimate that the value of α should not be too large. With these estimates at hand, we prove uniqueness by adapting an argument by Bando and Mabuchi in the Kähler–Einstein situation [8]. An interesting open question is to see what the largest value of α is, for which solutions exist. Notice that in the *dissolving limit* $\tau \to N/2$ of the vortex we have $\phi \to 0$ (see [23]), and α^* in (3.8) becomes arbitrarily large.

4 Non-abelian gravitating vortices

One can consider the dimensional reduction of the Kähler–Yang–Mills equations for higher rank $\mathrm{SU}(2)$-equivariant bundles on $X \times \mathbb{P}^1$, where X is a compact Riemann surface. In particular one can consider extensions of the form

$$0 \to p^* E_1 \longrightarrow E \longrightarrow p^* E_2 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(2) \to 0, \tag{4.1}$$

where E_1 and E_2 are holomorphic vector bundles on X and, as above, p and q are the projections from $X \times \mathbb{P}^1$ to X and $\mathbb{P}^1$ respectively. Extensions of the form (4.1) are in one-to-one correspondence with triples $T = (E_1, E_2, \phi)$, where ϕ is a sheaf homomorphism from E_2 to E_1, that is an element in $H^0(X, \mathrm{Hom}(E_2, E_1))$. As (3.2), these extensions define $\mathrm{SU}(2)$-equivariant (in fact, $\mathrm{SL}(2, \mathbb{C})$-equivariant) vector bundles over $X \times \mathbb{P}^1$.

Given a triple $T = (E_1, E_2, \phi)$ over X we can consider the *gravitating coupled vortex equations* for a metric on X with Kähler form ω, and Hermitian metrics h_1 and h_2 on E_1 and E_2, respectively, given by

$$i\Lambda_\omega F_{h_1} + \phi\phi^* = \tau_1, \qquad i\Lambda_\omega F_{h_2} - \phi^*\phi = \tau_2,$$
$$S_\omega + \alpha\Delta_\omega|\phi|^2 - \alpha\big(\Lambda_\omega^2 \operatorname{tr} F_{h_1}^2 + \Lambda_\omega^2 \operatorname{tr} F_{h_2}^2 + 4\tau_1 \operatorname{tr}(i\Lambda_\omega F_{h_1}) + 4\tau_2 \operatorname{tr}(i\Lambda_\omega F_{h_2})\big) = c. \qquad (4.2)$$

See Section 3 for the notations. Here τ_1 and τ_2 are real parameters (of which a certain linear combination is linked to the Chern classes of E_1 and E_2), so that $\tau_1 - \tau_2 > 0$, $\alpha \in \mathbb{R}$ is a coupling constant, and $c \in \mathbb{R}$ depends on α, $\tau_1 - \tau_2$, and the topology of X and E_1 and E_2.

Remark 4.1. When referring to the gravitating coupled vortex equations we are of course assuming that at least one of the two vector bundles has rank bigger than one. The case in which both vector bundles have rank one can be reduced to the study of the abelian gravitating vortex equation as shown in [1].

Let $\sigma := \tau_1 - \tau_2$, and consider now the $\mathrm{SU}(2)$-equivariant Kähler form

$$\omega_\sigma = \sigma p^*\omega + q^*\omega_{\mathrm{FS}},$$

where ω is a Kähler form on X and ω_{FS} is the Fubini–Study metric on $\mathbb{P}^1$ (see Section 3). Similarly to Proposition 3.1, one has the following [3].

Proposition 4.2. *Let $T = (E_1, E_2, \phi)$ be a triple over X. The pair (X, T) admits a solution (ω, h_1, h_2) of the gravitating coupled vortex equations (4.2) if and only if $(X \times \mathbb{P}^1, E)$ admits an $\mathrm{SU}(2)$-invariant solution of the Kähler–Yang–Mills equations (2.5) with Kähler form ω_σ.*

For a fixed Kähler metric ω, the first two equations in (4.2) are the *coupled vortex equations* introduced in [24], where it was shown that they are dimensional reduction of the Hermite–Yang–Mills equations. An existence theorem for the coupled vortex equations was given in [12] in terms of a certain notion of stability for the triple T depending on the parameter σ. To define this concept, let $T = (E_1, E_2, \phi)$ and $T' = (E_1', E_2', \phi')$ be two triples on X. A homomorphism from T' to T is a commutative diagram

$$
\begin{array}{ccc}
E_2' & \xrightarrow{\phi'} & E_1' \\
\downarrow & & \downarrow \\
E_2 & \xrightarrow{\phi} & E_1,
\end{array}
$$

where the vertical arrows are holomorphic maps. A triple $T' = (E_1', E_2', \phi')$ is a subtriple of $T = (E_1, E_2, \phi)$ if the sheaf homomorphims $E_1' \to E_1$ and $E_2' \to E_2$ are injective. A subtriple $T' \subset T$ is called *proper* if $T' \neq 0$ and $T' \neq T$.

For any $\sigma \in \mathbb{R}$ the σ-*degree* and σ-*slope* of T are defined to be

$$\deg_\sigma(T) = \deg(E_1) + \deg(E_2) + \sigma \operatorname{rk}(E_2),$$
$$\mu_\sigma(T) = \frac{\deg_\sigma(T)}{\operatorname{rk}(E_1) + \operatorname{rk}(E_2)} = \mu(E_1 \oplus E_2) + \sigma\frac{\operatorname{rk}(E_2)}{\operatorname{rk}(E_1) + \operatorname{rk}(E_2)},$$

where $\deg(E)$, $\operatorname{rk}(E)$ and $\mu(E) = \deg(E)/\operatorname{rk}(E)$ are the degree, rank and slope of E, respectively.

We say $T = (E_1, E_2, \phi)$ is σ-*stable* if $\mu_\sigma(T') < \mu_\sigma(T)$ for any proper subtriple $T' = (E_1', E_2', \phi')$. We define σ-*semi-stability* by replacing the above strict inequality with a weak inequality. A triple is called σ-*polystable* if it is the direct sum of σ-stable triples of the same σ-slope. We denote by $\mathcal{M}_\sigma = \mathcal{M}_\sigma(n_1, n_2, d_1, d_2)$ the moduli space of σ-polystable triples $T = (E_1, E_2, \phi)$ which have $\mathrm{rk}(E_i) = n_i$ and $\deg(E_i) = d_i$ for $i = 1, 2$.

There are certain necessary conditions in order for σ-semistable triples to exist. Let $\mu_i = d_i/n_i$ for $i = 1, 2$. We define

$$\sigma_m = \mu_1 - \mu_2, \qquad \sigma_M = \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right)(\mu_1 - \mu_2), \qquad n_1 \neq n_2.$$

Proposition 4.3 ([12, Theorem 6.1])**.** *The moduli space $\mathcal{M}_\sigma(n_1, n_2, d_1, d_2)$ is a complex analytic variety, which is projective when σ is rational. A necessary condition for $\mathcal{M}_\sigma(n_1, n_2, d_1, d_2)$ to be non-empty is*

$$0 \leq \sigma_m \leq \sigma \leq \sigma_M \quad if \quad n_1 \neq n_2, \qquad 0 \leq \sigma_m \leq \sigma \quad if \quad n_1 = n_2.$$

The non-emptiness and topology of these moduli spaces have been studied in [13, 26, 27]. Triples play an important role in the study of the moduli space of Higgs bundles and character varieties of the fundamental group of X [13, 14].

The study of existence of solutions to the gravitating coupled vortex equations (4.2) in the higher rank case is entirely open, to the knowledge of the author. Solving this problem will most likely require new analytic and algebraic tools and techniques, that may turn out to be very useful for the study of existence of solutions of the Kähler–Yang–Mills equations (2.4). A particularly interesting situation, similarly to the abelian case should be the case in which $X = \mathbb{P}^1$. In this situation, one may conjecture the following.

Conjecture 4.4. *Let $T = (E_1, E_2, \phi)$ be a triple over $\mathbb{P}^1$. The pair $(\mathbb{P}^1, T)$ admits a solution to the gravitating coupled vortex equations (4.2) if and only T is σ-polystable and the point $T \in \mathcal{M}_\sigma$ is GIT polystable for the natural action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathcal{M}_\sigma$ induced by the action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{P}^1$.*

Remark 4.5. When $n_1 = n_2 = 1$, σ-stability of the triple T reduces to the condition (3.4), where $L = E_1 \otimes E_2^*$ and σ is essentially the inverse of τ. Then, the proof of this conjecture reduces to Theorem 3.7.

Acknowledgements

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